1 Concave and convex functions

1.1 Convex Sets

Definition 1 A set $X \subset \mathbb{R}^n$ is called convex if given any two points $x', x'' \in X$ the line segment joining $x'$ and $x''$ completely belongs to $X$, in other words for each $t \in [0,1]$ the point

$$x_t = (1-t)x' + tx''$$

is also in $X$ for every $t \in [0,1]$.

The intersection of convex sets is convex.

The union of convex sets is not necessarily convex.

Let $X \subset \mathbb{R}^n$. The convex hull of $X$ is defined as the smallest convex set that contain $X$.

The convex hull of $X$ consists of all points which are convex combinations of some points of $X$

$$CH(X) = \{y \in \mathbb{R}^n : y = \sum t_ix_i, \ x_i \in X, \ \sum t_i = 1\}.$$  

1.2 Concave and Convex Function

A function $f$ is concave if the line segment joining any two points on the graph is never above the graph. More precisely

Definition 2 A function $f : S \subset \mathbb{R}^n \to \mathbb{R}$ defined on a convex set $S$ is concave if given any two points $x', x'' \in S$ we have

$$(1-t)f(x') + tf(x'') \leq f((1-t)x' + tx'')$$

for any $t \in [0,1]$.

$f$ is called strictly concave if

$$(1-t)f(x') + tf(x'') < f((1-t)x' + tx'').$$

Definition 3 A function $f : S \subset \mathbb{R}^n \to \mathbb{R}$ is convex if given any two points $x', x'' \in S$ we have

$$(1-t)f(x') + tf(x'') \geq f((1-t)x' + tx'')$$

for any $t \in [0,1]$.

$f$ is called strictly convex if

$$(1-t)f(x') + tf(x'') > f((1-t)x' + tx'').$$
Roughly speaking concavity of a function means that the graph is above chord.
It is clear that if $f$ is concave then $-f$ is convex and vice versa.

**Theorem 1** A function $f : S \subset R^n \rightarrow R$ is concave (convex) if and only if its restriction to every line segment of $R^n$ is concave (convex) function of one variable.

**Theorem 2** If $f$ is a concave (convex) function then a local maximizer (minimizer) is global.

1.2.1 Characterization in Terms of Graphs
Given a function $f : S \subset R^n \rightarrow R$ defined on a convex set $S$.

The hypograph of $f$ is defined as the set of points $(x,y) \in S \times R$ lying on or bellow the graph of the function:

$$\text{hyp } f = \{(x,y) : x \in S, y \leq f(x)\}.$$  

Similarly, the epigraph of $f$ is defined as the set of points $(x,y) \in S \times R$ lying on or above the graph of the function:

$$\text{epi } f = \{(x,y) : x \in S, y \geq f(x)\}.$$  

**Theorem 3** (a) A function $f : S \subset R^n \rightarrow R$ defined on a convex set $S$ is concave if and only if its hypograph hyp $f$ is convex.

(b) A function $f : S \subset R^n \rightarrow R$ defined on a convex set $S$ is convex if and only if its epigraph epi $f$ is convex.

**Proof of (a).** Let $(x_1,y_1),(x_2,y_2) \in \text{hyp } f$, let us show that

$$(xt,yt) = (tx_1 + (1-t)x_2, ty_1 + (1-t)y_2) \in \text{hyp } f.$$

$$y_t = ty_1 + (1-t)y_2 \leq tf(x_1) + (1-t)f(x_2) \leq f(tx_1 + (1-t)x_2) = f(x_t).$$

1.2.2 Characterization in Terms of Level Sets
Given a function $f : S \subset R^n \rightarrow R$ defined on a convex set $S$.

Take any number $K \in R$.

The upper contour set $U_K$ of $f$ is defined as

$$U_K = \{x \in S, f(x) \geq K\}.$$  

Similarly, the lower contour set $L_K$ of $f$ is defined as

$$L_K = \{x \in S, f(x) \leq K\}.$$  

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**Theorem 4** (a) Suppose a function $f : S \subset \mathbb{R}^n \to \mathbb{R}$ defined on a convex set $S$ is concave. Then for every $K$ the upper contour set $U_K$ is either empty or a convex set.

(b) If $f$ is convex, then for every $K$ the lower contour set $L_K$ is either empty or a convex set.

**Proof.** Let us prove only (a).

Let $x_1, x_2 \in U_k$, let us show that $x_t = tx_1 + (1-t)x_2 \in U_K$:

$f(x_t) = f(tx_1 + (1-t)x_2) \geq tf(x_1) + (1-t)f(x_2) \geq tK + (1-t)K = K.$

**Remark.** Notice that this is only necessary condition, not sufficient: consider the example $f(x) = e^x$ or $f(x) = x^3$.

### 1.2.3 Examples of Concave Functions

**Theorem 5** Suppose $f_1, ..., f_n$ are concave (convex) functions and $a_1 > 0, ..., a_n > 0$, then the linear combination

$$F = a_1 f_1 + ... + a_n f_n$$

is concave (convex).

**Proof.**

$$F((1 - t)x + ty) = \sum a_i f_i((1 - t)x + ty) \geq \sum a_i[(1 - t)f_i(x) + tf_i(y)] = (1 - t) \sum a_i f(x) + t \sum a_i f(y) = (1 - t)F(x) + tF(y).$$

A function of the form $f(x) = f(x_1, x_2, ..., x_n) = a_0 + a_1 x_1 + a_2 x_2 + ... + a_n x_n$ is called affine function (if $a_0 = 0$, it is a linear function).

**Theorem 6** An affine function is both concave and convex.

**Proof.** The theorem follows from previous theorem and following easy to prove statements:

1. The function $f(x_1, ..., x_n) = x_i$ is concave and convex;
2. The function $f(x_1, ..., x_n) = -x_i$ is concave and convex;
3. The constant function $f(x_1, ..., x_n) = a$ is concave and convex.

**Theorem 7** A concave monotonic transformation of a concave function is itself concave.

**Proof.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a concave function and $g : \mathbb{R} \to \mathbb{R}$ be concave and increasing, then

$$(g \circ f)(1 - t)x + ty = g(f((1 - t)x + ty)) \geq g((1 - t)f(x) + tf(y)) \geq (1 - t)g(f(x)) + tg(f(y)) = (1 - t)(g \circ f)(x) + t(g \circ f)(y).$$
here the first inequality holds since $f$ is concave and $g$ is increasing, and the second inequality holds since $g$ is concave.

**Remark.** Note that just monotonic transformation of a concave function is not necessarily concave: consider, for example $f(x) = x$ and $g(z) = z^3$.

Thus the *concavity of a function is not ordinal*, it is cardinal property.

**Economic Example**

Suppose production function $f(x)$ is *concave* and the cost function $c(x)$ is *convex*. Suppose also $p$ is the positive selling price. Then the profit function

$$\pi(x) = pf(x) + (-c(x))$$

is *concave* as a linear combination with positive coefficients of concave functions. Thus a local maximum of profit function is global in this case (see bellow).

### 1.3 Calculus Criteria for Concavity

For one variable functions we have the following statements

1. A $C^1$ function $f : U \subset \mathbb{R} \to \mathbb{R}$ is concave if and only if its first derivative $f'(x)$ is decreasing function.

2. A $C^2$ function $f : U \subset \mathbb{R} \to \mathbb{R}$ is concave if and only if its second derivative $f''(x)$ is $\leq 0$.

In $n$-variable case usually instead of $f'(x)$ we consider the Jacobian (gradient) $Df(x)$ and instead of $f''(x)$ we consider the hessian $D^2f(x)$.

It is not clear how to generalize the above statements 1 and 2 to $n$-variable case since the statement "$Df(x)$ (which is a vector) is decreasing function” has no sense as well as ”$D^2f(x)$ (which is a matrix) is positive”.

Let us reformulate the statements 1 and 2 in the following forms:

1'. A $C^1$ function $f : U \subset \mathbb{R} \to \mathbb{R}$ is concave if and only if

$$f(y) - f(x) \leq f'(x)(y - x)$$

for all $x, y \in U$.

Hint: Observe that for concave $f(x)$ and $x < y$ one has

$$f'(x) \geq \frac{f(y) - f(x)}{y - x} \geq f'(y).$$

2'. A $C^2$ function $f : U \subset \mathbb{R} \to \mathbb{R}$ is concave if and only if the one variable quadratic form $Q(y) = f''(x) \cdot y^2$ is negative semidefinite for all $x \in U$.

Hint: Observe that the quadratic form $Q(y) = f''(x) \cdot y^2$ is negative semidefinite if and only if the coefficient $f''(x) \leq 0$.  

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Now we can formulate the multi-variable generalization of 1:

**Theorem 8** A $C^1$ function $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$ is concave if and only if

$$f(y) - f(x) \leq Df(x)(y - x),$$

for all $x, y \in U$, that is

$$f(y) - f(x) \leq \frac{\partial f}{\partial x_1}(x)(y_1 - x_1) + \ldots + \frac{\partial f}{\partial x_n}(x)(y_n - x_n).$$

Similarly $f$ is convex if and only if

$$f(y) - f(x) \geq Df(x)(y - x).$$

Remember that concavity of a function means that the graph is above chord? Now we can say

Roughly speaking concavity of a function means that the tangent is above graph.

From this theorem follows

**Corollary 1** Suppose $f$ is concave and for some $x_0, y \in U$ we have

$$Df(x_0)(y - x_0) \leq 0,$$

then $f(y) \leq f(x_0)$ for THIS $y$.

Particularly, if directional derivative of $f$ at $x_0$ in any feasible direction is nonpositive, i.e.

$$D_{y-x_0}f(x_0) = Df(x_0)(y - x_0) \leq 0$$

for ALL $y \in U$, then $x_0$ is GLOBAL max of $f$ in $U$.

Indeed, since of concavity of $f$ we have

$$f(y) - f(x_0) \leq Df(x_0)(y - x_0) \leq 0.$$

The following theorem is the generalization of 2:

**Theorem 9** A $C^2$ function $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$ defined on a convex open set $U$ is

(a) concave if and only if the Hessian matrix $D^2 f(x)$ is negative semidefinite for all $x \in U$;

(b) strictly concave if the Hessian matrix $D^2 f(x)$ is negative definite for all $x \in U$;

(c) convex if and only if the Hessian matrix $D^2 f(x)$ is positive semidefinite for all $x \in U$;

(d) strictly convex if the Hessian matrix $D^2 f(x)$ is positive definite for all $x \in U$;
Remark. Note that the statement (b) (and (d) too) is not "only if": If \( f \) is strictly concave then the Hessian is not necessarily negative definite for ANY \( x \). Analyze, for example \( f(x) = -x^4 \).

Let us recall criteria for definiteness of matrix in terms of principal minors:

1. A matrix \( H \) is positive definite if and only if its \( n \) leading principal minors are \( > 0 \).

2. A matrix \( H \) is negative definite if and only if its \( n \) leading principal minors alternate in sign so that all odd order ones are \( < 0 \) and all even order ones are \( > 0 \).

3. A matrix \( H \) is positive semidefinite if and only if its \( 2^n - 1 \) principal minors are all \( \geq 0 \).

4. A matrix \( H \) is negative semidefinite if and only if its \( 2^n - 1 \) principal minors alternate in sign so that odd order minors are \( \leq 0 \) and even order minors are \( \geq 0 \).

Example. Let us determine the definiteness of the matrix \( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \).

Its first order principal minors are

\[ M_1 = 1, \quad M'_1 = 0, \]

and the only second order principal minor is

\[ M_2 = 0. \]

We are in the situation (3), so our matrix is positive semidefinite. Note that corresponding quadratic form is \( Q(x, y) = y^2 \).

Example. Let \( f(x, y) = 2x - y - x^2 + 2xy - y^2 \). Its Hessian is

\[ \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \]

which is constant (does not depend on \((x, y)\)) and negative semidefinite. Thus \( f \) is concave.

Example. Consider the function \( f(x, y) = 2xy \). Its Hessian is

\[ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

Since the only second order principal minor is \(-1 < 0\) the matrix is indefinite, thus \( f \) is neither concave nor convex.
**Example.** Consider the Cobb-Douglas function \( f(x, y) = cx^a y^b \) with \( a, b, c > 0 \) in the first orthant \( x > 0, y > 0 \).

Its hessian is

\[
\begin{pmatrix}
    a(a-1)cx^{a-2}y^b & abcx^{a-1}y^{b-1} \\
    abcx^{a-1}y^{b-1} & b(b-1)cx^ay^{b-2}
\end{pmatrix}.
\]

The principal minors of order 1 of this matrix are

\[
M_1 = a(a-1)cx^{a-2}y^b, \quad M'_1 = b(b-1)cx^ay^{b-2}
\]

and the only principal minor of order 2 is

\[
M_2 = abcx^{2a-2}y^{2b-2}(1-(a+b)).
\]

**When this function is concave?** For this the Hessian must be negative semidefinite. This happens when all principal minors of degree 1 \( M_1 \) and \( M'_1 \) are \( \leq 0 \) and (only) principal minor of degree 2 \( M_2 \) is \( \geq 0 \).

Recall that we work in the first orthant \( x > 0, y > 0, \) and \( a, b, c > 0 \).

If our \( f(x, y) = cx^ay^b \) exhibits constant or decreasing return to scale (CRS or DRS), that is \( a+b \leq 1 \), then clearly \( a \leq 0, b \leq 0 \), and we have thus the Cobb-Douglas function is concave if and only if \( M_1 \leq 0, M'_1 \leq 0, M_2 \geq 0 \), thus \( f \) is concave.

**Remark.** So we have shown that if a Cobb-Douglas function \( f(x, y) = cx^ay^b \) is CRS or DRS, it is concave. But can it be convex?

### 1.4 Concave Functions and Optimization

Concavity of a function replaces the second derivative test to separate local max, min or saddle, moreover, for a concave function a critical point which is local max (min) is global:

**Theorem 10** Let \( f : U \subset \mathbb{R}^n \rightarrow \mathbb{R} \) be concave (convex) function defined on a convex open set \( U \). If \( x^* \) is a critical point, that is \( Df(x^*) = 0 \), then it is global maximizer (minimizer).

**Proof.** Since \( Df(x^*) = 0 \) from the inequality

\[
f(y) - f(x^*) \leq Df(x^*)(y - x^*) = 0
\]

follows \( f(y) \leq f(x^*) \) for all \( y \in U \).

The next result is stronger, it allows to find maximizer also on the boundary of \( U \) if it is not assumed open:

**Theorem 11** Let \( f : U \subset \mathbb{R}^n \rightarrow \mathbb{R} \) be concave function defined on a convex set \( U \). If \( x^* \) is a point, which satisfies

\[
Df(x^*)(y - x^*) \leq 0
\]
for each $y \in U$, then $x^*$ is a global maximizer of $f$ on $U$.

Similarly, if $f$ is convex and

$$Df(x^*)(y - x^*) \geq 0$$

for each $y \in U$, then $x^*$ is a global minimizer of $f$ on $U$.

**Proof.** From

$$f(y) - f(x^*) \leq Df(x^*)(y - x^*) \leq 0$$

follows $f(y) \leq f(x^*)$ for all $y \in U$.

**Remark.** Here is an example of global maximizer which is not a critical point: Suppose $f : R \rightarrow R$ is an increasing and convex function on $[a, b]$. Then $f'(b)(x - b) \leq 0$ for all $x \in [a, b]$. Thus $b$ is global maximizer of $f$ on $[a, b]$.

**Lagrange Case**

Consider the problem

$$\max f(x_1, ..., x_n) \text{ s.t. } h_i(x) = c_i, \ i = 1, ..., k.$$ 

As we know if $x^* = (x_1^*, ..., x_n^*)$ is a maximizer, then there exist $\mu^* = (\mu_1^*, ..., \mu_k^*)$ such that $(x^*, \mu^*)$ satisfies Lagrange conditions $Df(x^*) - \mu^* \cdot Dh(x^*) = 0$ and $h_i(x^*) = c_i, \ i = 1, ..., k$.

This is the sufficient condition for a global maximum:

**Theorem 12** Suppose $f$ is concave, each $h_i$ is convex, $(x^*, \mu^*)$ satisfies Lagrange conditions and each $\mu_i \geq 0$. Then $x^*$ is a global maximizer.

**KKT Case**

Consider the problem

$$\max f(x_1, ..., x_n) \text{ s.t. } g_i(x) \leq c_i, \ i = 1, ..., k.$$ 

As we know if $x^* = (x_1^*, ..., x_n^*)$ is a maximizer, then there exist $\lambda^* = (\lambda_1^*, ..., \lambda_k^*)$ such that $(x^*, \lambda^*)$ satisfies KKT conditions $Df(x^*) - \lambda^* \cdot Dg(x^*) = 0, \ \lambda_i \cdot (h_i(x^* - c_i) = 0, \ i = 1, ..., k, \ \lambda_i \geq 0, \ g_i(x^*) = c_i, \ i = 1, 2, ..., k$.

This is the sufficient condition for a global maximum:

**Theorem 13** Suppose $f$ is concave, each $g_i$ is convex, and $(x^*, \lambda^*)$ satisfies KKT conditions. Then $x^*$ is a global maximizer.
Example. Consider a production function \( y = g(x_1, ..., x_n) \), where \( y \) denotes output, \( x = (x_1, ..., x_n) \) denotes the input bundle, \( p \) denotes the price of output and \( w_i \) is the cost per unit of input \( i \). Then the cost function is

\[
C(x) = w_1 x_1 + ... + w_n x_n,
\]

and the profit function is

\[
\pi(x) = pg(x) - C(x).
\]

Our first claim is that if \( g \) is concave, then \( \pi \) is concave too: \( C(x) \), as a linear function, is convex, then \( -C(x) \) is concave, besides \( pg(x) \) is concave too since \( p > 0 \), then \( \pi(x) = pg(x) + (-C(x)) \) is concave.

The first order condition gives

\[
\frac{\partial \pi(x)}{\partial x_i} = p \frac{\partial g(x)}{\partial x_i} - w_i = 0.
\]

Since of concavity this condition is necessary and sufficient to be interior maximizer. This means that the maximizer of profit is the value of \( x \) where marginal revenue product \( p \frac{\partial g(x)}{\partial x_i} \) equals to the factor price \( w_i \) for each input.

1.5 Quasiconcave Functions

Recall the property of a concave function \( f \): for each \( K \) the lower level set

\[
L_K = \{x, f(x) \leq K\}
\]

is concave.

This property is taken as the definition of quasiconcave function:

**Definition 1.** A function \( f(x) \) defined on a convex subset \( U \subset \mathbb{R}^n \) is quasi-concave if

\[
L_K = \{x : f(x) \leq K\}
\]

is a convex set for any constant \( K \).

Similarly, \( f \) is quasiconvex if

\[
U_K = \{x : f(x) \geq K\}
\]

is a convex set for any constant \( K \).

**Definition 2.** A function \( f(x) \) defined on a convex subset \( U \subset \mathbb{R}^n \) is quasi-concave if

\[
f(tx + (1-t)y) \geq \min(f(x), f(y))
\]

for each \( x, y \in U \) and \( t \in [0,1] \).
Similarly, \( f \) is quasiconvex if
\[
 f(tx + (1 - t)y) \leq \max(f(x), f(y)).
\]

**Remark.** Concavity implies, but is not implied by quasiconcavity. Indeed, the function \( f(x) = x^3 \) is quasiconcave (and quasiconvex) but not concave (and convex).

**Remark.** Besides \( f \) is quasiconcave \( f \) and only if \(-f\) is quasiconvex.

**Theorem 14** Definition 1 and Definition 2 are equivalent.

**Proof.** (a) Def. 1 \( \Rightarrow \) Def. 2.

**Given:**
\[
 U_K = \{ x, \ f(x) \geq K \}
\]
is a convex set.

**Prove:**
\[
 f(tx + (1 - t)y) \geq \min(f(x), f(y)).
\]
Indeed, take \( K = \min(f(x), f(y)) \), suppose this min is \( f(x) \). Then \( K = f(x) \leq f(x) \), so \( x \in U_K \), and \( K = f(x) \leq f(y) \), so \( y \in U_K \). Then, since of convexity of \( U_K \) we have \( tx + (1 - t)y \in U_K \), that is \( K \leq f(tx + (1 - t)y) \).

(b) Def. 2 \( \Rightarrow \) Def. 1.

**Given:**
\[
 f(tx + (1 - t)y) \geq \min(f(x), f(y)).
\]

**Prove:**
\[
 U_K = \{ x, \ f(x) \geq K \}
\]
is a convex set.
Indeed, suppose \( x, y \in U_K \), that is \( f(x) \geq K, f(y) \geq y \). We want to prove that \( f(tx + (1 - t)y) \in U_K \), i.e. \( f(tx + (1 - t)y) \geq K \). Indeed, assume \( \min(f(x), f(y)) = f(x) \), then
\[
 f(tx + (1 - t)y \geq \min(f(x), f(y)) = f(x) \geq K.
\]

**Theorem 15** A monotonic transformation \( gf \) of a quasiconcave function \( f \) is itself quasiconcave.

**Proof.** Take any \( K \in R \). Since \( g \) is monotonic, there exists \( K' \in R \) such that \( K = g(K') \). Then
\[
 U_K(gf) = \{ x, \ gf(x) \geq K \} = \{ x, \ gf \geq g(K') \} = \{ x, \ f(x) \geq K' \} = U_{K'}(f) \]
is a convex set.

Remark. Thus the quasiconcavity is ordinal property (recall, the concavity is cardinal: a monotonic transformation of concave is not necessarily concave, for example \( f(x) = x \) is concave, \( g(x) = x^3 \) is monotonically increasing, but \( g(f(x)) = x^3 \) is not concave).

In particular a monotonic transformation of concave is quaziconcave. But there exists quaziconcave function which is not monotonic transformation of a concave function.

Example. Every Cobb-Douglas function \( F(x_1, x_2) = Ax_1^p x_2^q, \; p, q > 0 \) is quasiconcave:

(a) As we know an DRS (Decreasing Return to Scale) Cobb-Douglas function such as \( f(x_1, x_2) = x_1^{1/3} x_2^{1/3} \) concave.

(b) An IRS (Increasing Return to Scale) Cobb-Douglas function, such as \( x_1^{2/3} x_2^{2/3} \) is quasiconcave. Indeed, IRS Cobb-Douglas is monotonic transformation of DRS Cobb-Douglas:

\[
    x_1^{2/3} x_2^{2/3} = (x_1^{1/3} x_2^{1/3})^2,
\]

so \( x_1^{2/3} x_2^{2/3} = g(f(x_1, x_2)) \) where \( f(x_1, x_2) = x_1^{1/3} x_2^{2/3} \) and \( g(z) = z^2 \).

Example. Any CES function \( Q(x, y) = (ax^r + by^r)^\frac{1}{r}, \; a, b > 0, \; 0 < r < 1 \) is quasiconcave: \( Q(x, y) = gq(x, y) \) where \( q(x, y) = (ax^r + by^r) \) is a concave function because it is positive linear combination of concave functions, and \( q(z) = z^{\frac{1}{r}} \) is monotonic transformation.

Example. Any increasing function \( f : R \to R \) is quasiconcave (and quasiconvex):

\[
    U_K = \{ x, \; f(x) \geq K \} = [f^{-1}K, +\infty)
\]
is a convex set.

Example. Each function \( f : R^1 \to R^1 \) which monotonically rises until it reaches a global maximum and the monotonically decrease, such as \( f(x) = -x^2 \), is quasiconcave: \( U_K \) is convex.

1.5.1 Calculus Criterion for Quasiconcavity

\( F \) is quasiconcave if and only if

\[
    F(y) \geq F(x) \implies DF(x)(y - x) \geq 0.
\]

\( F \) is quasiconvex if and only if

\[
    F(y) \leq F(x) \implies DF(x)(y - x) \geq 0.
\]
Exercises

1. By drawing diagrams, determine which of the following sets is convex.

   (a) \{(x, y) : y = e^x\}.  (b) \{(x, y) : y \geq e^x\}.  (c) \{(x, y) : xy \geq 1, x > 0, y > 0\}.

2. Determine the definiteness of the following symmetric matrices

   \[
   \begin{pmatrix}
   0 & 0 \\
   0 & 0 \\
   \end{pmatrix},
   \begin{pmatrix}
   1 & 0 \\
   0 & 0 \\
   \end{pmatrix},
   \begin{pmatrix}
   0 & 1 \\
   0 & 0 \\
   \end{pmatrix},
   \begin{pmatrix}
   1 & 1 \\
   1 & 0 \\
   \end{pmatrix},
   \begin{pmatrix}
   0 & 0 \\
   0 & 1 \\
   \end{pmatrix},
   \begin{pmatrix}
   1 & 1 \\
   1 & 1 \\
   \end{pmatrix},
   \begin{pmatrix}
   1 & 0 \\
   0 & 1 \\
   \end{pmatrix},
   \begin{pmatrix}
   1 & 1 \\
   1 & 1 \\
   \end{pmatrix}
   \]

3. For each of the following functions, determine which, if any, of the following conditions the function satisfies: concavity, strict concavity, convexity, strict convexity. (Use whatever technique is most appropriate for each case.)

   (a) \( f(x, y) = x + y \)
   (b) \( f(x, y) = x^2 \)
   (c) \( f(x, y) = x + y - e^x - e^{x+y} \)
   (d) \( f(x, y, z) = x^2 + y^2 + 3z^2 - xy + 2xz + yz \)
   (e) \( f(x, y) = 3e^x + 5x^4 - ln x \)
   (f) \( f(x, y, z) = Ax^a y^b z^c, \ a, b, c > 0. \)

4. Let \( f(x_1, x_2) = x_1^2 - x_1 x_2 + x_2^2 + 3x_1 - 2x_2 + 1. \) Is \( f \) convex, concave, or neither?

5. Prove that any homogenous function on \((0, +\infty)\) is either concave or convex.

6. Suppose that a firm that uses 2 inputs has the production function \( f(x_1, x_2) = 12x_1^{1/3}x_2^{1/2} \) and faces the input prices \((p_1, p_2)\) and the output price \(q\). Show that \( f \) is concave for \( x_1 > 0 \) and \( x_2 > 0 \), so that the firm’s profit is concave.

7. Let \( f(x_1, x_2) = x_1^3 + 2x_1^2 + 2x_1 x_2 + (1/2)x_2^2 - 8x_1 - 2x_2 - 8. \) Find the range of values of \((x_1, x_2)\) for which \( f \) is convex, if any.

8. Determine the values of \( a \) (if any) for which the function

\[
2x^2 + 2xz + 2ayz + 2z^2
\]
is concave and the values for which it is convex.

9. Show that the function \( f(w, x, y, z) = -w^2 + 2wx - x^2 - y^2 + 4yz - z^2 \) is not concave.

**Homework**

Exercise 21.2c from [Simon], Exercise 21.12 from [Simon], Exercise 21.18 from [Simon], Exercise 3f, Exercise 6.
**Short Summary**

**Concave and Convex**

Convex set $X \subset \mathbb{R}^n$: $x', x'' \in X \Rightarrow x^t = (1-t)x' + tx'' \in X$.

Convex hull $CH(X) = \{y \in \mathbb{R}^n : y = \sum t_i x_i, \ x_i \in X, \ \sum t_i = 1\}$.

Convex function $f : S \subset \mathbb{R}^n \to \mathbb{R}$:
$x', x'' \in S \Rightarrow (1-t)f(x') + tf(x'') \leq f((1-t)x' + tx'')$, i.e. graph is above chord.

**Hypograph**: $\text{hyp } f = \{(x, y) : x \in S, \ y \leq f(x)\}$. $f$ is concave iff hyp $f$ is convex.

**Epigraph**: $\text{epi } f = \{(x, y) : x \in S, \ y \geq f(x)\}$. $f$ is convex iff epi $f$ is convex.

**Upper contour set**: $U_K = \{x \in S, \ f(x) \geq K\}$. If $f$ is concave then $U_K$ is convex.

**Lower contour set**: $U_K = \{x \in S, \ f(x) \leq K\}$. If $f$ is convex then $U_L$ is convex.

**Calculus Criteria**

$C^1$ function $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is concave iff $f(y) - f(x) \leq Df(x)(y - x)$.

$C^2$ function $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is concave iff $D^2f(x) \leq 0$.

**Concavity and Optimization**

If $f$ is concave and $D(x^*) = 0$ then $x^*$ is global max.
If $f$ is concave and $Df(x^*)(y - x^*) \leq 0$ for $\forall y$ then $x^*$ is global max.

**Quaziconcavity**

$f$ quasiconcave if $U_K = \{x : f(x) \geq K\}, \ \forall K$. Equivalently

$$f(tx + (1-t)y) \geq \min(f(x), f(y)), \ \forall \ x, y, \ t \in [0,1].$$

Concavity - cardinal, quasiconcavity - ordinal.

**Calculus Criterion**

$F$ is quasiconcave iff

$$F(y) \geq F(x) \ \Rightarrow \ \ DF(x)(y - x) \geq 0.$$