Lattices and Topology

Exercises for Lecture 5

1*. Let $(X, \leqslant, \tau)$ be a Priestley space. Prove that each open upset of $(X, \leqslant, \tau)$ is a union of clopen upsets of $(X, \leqslant, \tau)$. By a dual argument, show that each open downset is a union of clopen downsets.

2*. Let L be a bounded distributive lattice, $\mathscr{X}(L)$ be the set of prime filters of L, $\tau_{\subseteq}$ be the Alexandroff topology, τ_P be the Priestley topology, and τ_S be the spectral topology on $\mathscr{X}(L)$. Prove that $\tau_S = \tau_{\subseteq} \cap \tau_P$.

3. Show that for all elements a_1, a_2, b_1, b_2, c of a distributive lattice L,

$$a_1 \leqslant b_1 \vee c \text{ and } c \wedge a_2 \leqslant b_2$$

implies

$$a_1 \wedge a_2 \leqslant b_1 \vee b_2.$$

4*. Let $\mathscr{L}_{\mathrm{IPC}}$ be the set of provable equivalence classes of formulæ of the Intuitionistic Propositional Calculus IPC. Denote the equivalence class of a formula φ by $[\varphi]$. Define $\leqslant$ on $\mathscr{L}_{\mathrm{IPC}}$ by $[\varphi] \leqslant [\psi]$ iff $\varphi \vdash \psi$ is derivable in IPC.

4a. Show that $\leqslant$ is a well-defined partial order.

4b. Show that $(\mathscr{L}_{\mathrm{IPC}}, \leqslant)$ is a distributive lattice where $[\varphi] \vee [\psi] = [\varphi \vee \psi]$ and $[\varphi] \wedge [\psi] = [\varphi \wedge \psi]$.

4c. Prove that $(\mathscr{L}_{\mathrm{IPC}}, \leqslant)$ is a Heyting lattice, where $[\varphi] \to [\psi] = [\varphi \to \psi]$.

4d. Prove that if we replace IPC by the Classical Propositional Calculus CPC, then $(\mathscr{L}_{\mathrm{CPC}}, \leqslant)$ is a Boolean lattice.

5. Let P be a poset and $\mathscr{U}(P)$ be the Heyting lattice of upsets of P. For $U, V \in \mathscr{U}(P)$ show that

$$U \to V = \{w \in P : \text{ for all } w' \geqslant w, \text{ if } w' \in U, \text{ then } w' \in V\}.$$

6. Let (X, τ) be a topological space. Show that in the Heyting lattice of open subsets of (X, τ) the implication is given by

$$U \to V = \mathrm{int}((X - U) \cup V).$$