Lattices and Topology

Exercises for Lecture 4

1. Let $(X, \leqslant, \tau)$ be a Priestley space.
 1a. Prove that (X, τ) is Hausdorff.
 1b. Prove that (X, τ) is zero-dimensional.

2. Let $(X, \leqslant, \tau)$ be a Priestley space. Recall that $\psi : X \to X^*{}_*$ is given by

$$\psi(x) = \{U \in X^* : x \in U\}.$$

 2a. Show that ψ is well-defined.
 2b. Show that ψ is 1-1.
 2c*. Prove that ψ is onto.
 2d*. Prove that ψ is continuous.
 2e. Conclude that ψ is an order-isomorphism and a homeomorphism.

3. Derive the Birkhoff duality from the Priestley duality.

4. Let B be a Boolean lattice.
 4a. For two prime filters x and y of B, show that if $x \subseteq y$ then $x = y$.
 4b. Conclude that the Stone duality follows from the Priestley duality.

5*. Let L be a Heyting lattice. Prove that for all $a, b \in L$ we have:

$$\phi(a \to b) = \mathscr{X}(L) - {\downarrow}[\phi(a) - \phi(b)].$$

6. Let $(X, \leqslant, \tau)$ be an Esakia space. For two clopen upsets U, V of $(X, \leqslant, \tau)$, set

$$U \to V = X - {\downarrow}(U - V).$$

Show that the lattice of clopen upsets of $(X, \leqslant, \tau)$ together with this operation is a Heyting lattice.

7. Give an example of a Priestley space which is not an Esakia space.