Lattices and Topology

Exercises for Lecture 1

1. Give a detailed proof that all nonempty finite subsets of a lattice possess suprema and infima.

2. Show that each complete lattice is bounded.

3. Prove that in a lattice the operations $a \lor b = \operatorname{Sup}\{a, b\}$ and $a \land b = \operatorname{Inf}\{a, b\}$ satisfy commutativity, associativity, and absorption laws.

4. Let $\land, \lor : L \times L \to L$ be binary operations on a set $L$ satisfying commutativity, associativity, idempotency, and absorption laws. Define the binary relation $\leq$ on $L$ by
   
   $$a \leq b \iff a \land b = a.$$  

4a. Prove that $a \leq b \iff a \lor b = a$.

4b. Prove that $\leq$ is a partial order.

4c. Prove that $a \land b = \operatorname{Inf}\{a, b\}$ and $a \lor b = \operatorname{Sup}\{a, b\}$.

5. Prove that the inverse of a lattice isomorphism is a lattice isomorphism.

6. Let $L$ be a lattice. Prove that for each $a, b, c \in L$ we have:

   $$(a \land b) \lor (a \land c) \leq a \land (b \lor c) \quad \text{and} \quad a \lor (b \land c) \leq (a \lor b) \land (a \lor c).$$

7. Prove that each linearly ordered set is a distributive lattice.

8. Show that in a bounded linearly ordered set the only elements possessing a complement are 0 and 1.

9. Prove that every finite distributive lattice is a Heyting lattice.

10*. Prove that a complete lattice $L$ is a Heyting lattice iff the following infinite distributive law

   $$a \land \bigvee S = \bigvee\{a \land s : s \in S\}$$

holds in $L$ for all $a \in L$ and $S \subseteq L$. 