Lattices and Topology

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Lecture 5: Applications to logic

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- We saw how the Priestley duality results in the representation of a bounded distributive lattice as the lattice of clopen upsets of a Priestley space.

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Definition: We call a space spectral if it is compact, coherent, and sober.

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In particular, this implies the following representation of bounded distributive lattices.

Topological representation theorem: Each bounded distributive lattice is isomorphic to a sublattice of τ_S. Therefore each bounded distributive lattice can be represented as a sublattice of the lattice of open subsets of some topological space.

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Thus we can develop a duality for distributive lattices by means of either topology and order—Priestley duality—where topology behaves rather nicely; or only by means of topology—spectral duality—but then the topology is not as nice as in the other case.

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G. Bezhanishvili, N. Bezhanishvili, D. Gabelaia, A. Kurz. *Bitopological duality for distributive lattices and Heyting algebras*, available at `http://www.cs.le.ac.uk/people/nb118/Publications/PairwiseStone.pdf`

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A proof in each of the systems consists of a succession of sequents each of which is derivable from the previous ones according to the inference rules.

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Moreover, conjunction will be interpreted by meet, disjunction by join, and implication by the Heyting implication in case of IPC and by the Boolean implication in case of CPC.

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$v(\varphi \to \psi) = v(\varphi) \to v(\psi)$.

Algebraic semantics

A sequent $\varphi_1, \ldots, \varphi_m \vdash \psi_1, \ldots, \psi_n$ is said to be true in such a model if

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In principle the only thing we need to know about the inference rules is that they ensure soundness of the corresponding system with respect to the semantics.

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Identifying provably equivalent formulæ one obtains a lattice of appropriate type equipped with the valuation v which assigns to a formula φ its equivalence class.

In this way, we obtain a model, and it is then not difficult to see that a sequent is derivable iff it is true in this model.

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Our first representation theorem of Lecture 2 implies that each bounded distributive lattice L is isomorphic to a sublattice of the lattice $\mathscr{U}(P)$ of upsets of some poset P.

This theorem implies that the implication-free fragment of IPC is complete with respect to the relational semantics—the semantics in which the only models allowed are those in which formulæ are interpreted as upsets of a poset P, the conjunction as set-theoretic intersection, and the disjunction as set-theoretic union.

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For those familiar with Kripke semantics of IPC, the above completeness is just a reformulation of the Kripke completeness of IPC. Put differently, Kripke completeness of IPC is nothing more but a representation of Heyting lattices as lattices of upsets of posets!

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On the other hand, IPC is complete with respect to an infinite class of finite models—another famous result from the thirties by Stanislaw Jaśkowski.

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- We have developed the Birkhoff duality between finite distributive lattices and finite posets.

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- As a result we have obtained relational and topological representations of bounded distributive lattices, Heyting lattices, and Boolean lattices.
- We have given applications of these representation theorems to logic. In particular, we have discussed several relational and topological completeness theorems for the intuitionistic and classical logics, and their implication-free fragments.

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