Lattices and Topology

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Lecture 5: Applications to logic
Review of lecture 4

We have described the Priestley topology on the set of prime filters of a bounded distributive lattice. We have defined Priestley spaces as ordered Stone spaces satisfying the Priestley separation axiom. We showed that the space of prime filters of a bounded distributive lattice is a Priestley space. We described the resulting Priestley duality between bounded distributive lattices and Priestley spaces. We saw how the Priestley duality results in the representation of a bounded distributive lattice as the lattice of clopen upsets of a Priestley space.
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- We showed that the space of prime filters of a bounded distributive lattice is a Priestley space.
- We described the resulting **Priestley duality** between bounded distributive lattices and Priestley spaces.
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We showed that the space of prime filters of a bounded distributive lattice is a Priestley space.

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We saw how the Priestley duality results in the representation of a bounded distributive lattice as the lattice of clopen upsets of a Priestley space.
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- We introduced **Esakia spaces** and obtained the **Esakia duality** between Heyting lattices and Esakia spaces from the Priestley duality.
- We saw how the Esakia duality gives representation of Heyting lattices as lattices of clopen upsets of Esakia spaces.
Short outline of lecture 4

- Spectral duality
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- Distributive lattices in logic
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This is how it was done originally by Marshall Stone back in 1937. For some this is the most natural way to define topology on the dual of $L$. 

Spectral topology
The Priestley topology as the patch topology

We look at the set $S = \{ \phi(a) : a \in L \}$. 

Since $\phi(0) = \emptyset$, $\phi(1) = X(L)$, and $\phi(a \land b) = \phi(a) \cap \phi(b)$, $S$ contains $\emptyset$, $X(L)$ and is closed under finite intersections. In addition, as $\phi(a \lor b) = \phi(a) \cup \phi(b)$, $S$ is closed under finite unions. But in general $S$ is not closed under arbitrary unions. Thus it does not form a topology.

We generate a topology from $S$ by closing $S$ under arbitrary unions. We call the obtained topology the spectral topology and denote it by $\tau_S$. 
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We look at the set \( S = \{ \phi(a) : a \in L \} \).

Since \( \phi(0) = \emptyset \), \( \phi(1) = \mathcal{K}(L) \), and \( \phi(a \wedge b) = \phi(a) \cap \phi(b) \),
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Thus we have two topologies on $\mathcal{H}(L)$—the Priestley topology $\tau_P$ and the spectral topology $\tau_S$. 
The Priestley topology as the patch topology

Thus we have two topologies on $\mathcal{X}(L)$—the Priestley topology $\tau_p$ and the spectral topology $\tau_S$. How are these two topologies related to each other?

We recall that $\tau_p$ is generated by $B = \{\phi(a) - \phi(b) : a, b \in L\}$ and $\tau_S$ is generated by $S = \{\phi(a) : a \in L\}$. It follows at once that $\tau_S$ is a subtopology of $\tau_p$, that is $\tau_S \subseteq \tau_p$. But more is true. In fact, each element of $B$ is the intersection of an element of $S$ and set-theoretic complement of an element of $S$. Thus $\tau_p$ is generated by the set $\{U \cap F : U \in S$ and $X(L) - F \in S\}$. When one topology is obtained from another this way, it is known in the literature as the patch topology. Therefore $\tau_p$ is the patch topology of $\tau_S$. 
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When one topology is obtained from another this way, it is known in the literature as the patch topology. Therefore $\tau_P$ is the patch topology of $\tau_S$. 
Consequently we can recover the Priestley topology $\tau_P$ from the spectral topology $\tau_S$ by taking the patch topology of $\tau_S$. 

Lemma: $\subseteq$ is the specialization order of $(X(L), \tau_S)$. 

Proof: For two prime filters $x$, $y$, we have $x \subseteq y$ iff $(\forall a \in L)(a \in x$ implies $a \in y)$. Therefore $x \subseteq y$ iff $(\forall U \in \tau_S)(x \in U$ implies $y \in U)$. Thus $\subseteq$ is the specialization order of $(X(L), \tau_S)$. 
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**Lemma:** $\subseteq$ is the specialization order of $(\mathcal{X}(L), \tau_S)$.

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Since $S$ generates $\tau_S$, it follows that
$x \subseteq y$ iff $(\forall U \in \tau_S)(x \in U$ implies $y \in U)$.

Thus $\subseteq$ is the specialization order of $(\mathcal{X}(L), \tau_S)$. 
From Priestley topology to spectral topology

Consequently, in the Priestley space \((\mathcal{X}(L), \subseteq, \tau_P)\), \(\tau_P\) is the patch topology of \(\tau_S\).
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Consequently, in the Priestley space \((\mathcal{H}(L), \subseteq, \tau_P)\), \(\tau_P\) is the patch topology of \(\tau_S\) and \(\subseteq\) is the specialization order of \(\tau_S\).
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Consequently, in the Priestley space \((\mathcal{X}(L), \subseteq, \tau_P)\), \(\tau_P\) is the patch topology of \(\tau_S\) and \(\subseteq\) is the specialization order of \(\tau_S\). Thus we can recover the Priestley space \((\mathcal{X}(L), \subseteq, \tau_P)\) from the space \((\mathcal{X}(L), \tau_S)\).
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How do we go the other way around?
Consequently, in the Priestley space \((\mathcal{X}(L), \subseteq, \tau_P)\), \(\tau_P\) is the patch topology of \(\tau_S\) and \(\subseteq\) is the specialization order of \(\tau_S\). Thus we can recover the Priestley space \((\mathcal{X}(L), \subseteq, \tau_P)\) from the space \((\mathcal{X}(L), \tau_S)\).

How do we go the other way around? That is, how do we get \(\tau_S\) from \((\mathcal{X}(L), \subseteq, \tau_P)\)?
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Consequently, in the Priestley space \((\mathcal{X}(L), \subseteq, \tau_P)\), \(\tau_P\) is the patch topology of \(\tau_S\) and \(\subseteq\) is the specialization order of \(\tau_S\). Thus we can recover the Priestley space \((\mathcal{X}(L), \subseteq, \tau_P)\) from the space \((\mathcal{X}(L), \tau_S)\).

How do we go the other way around? That is, how do we get \(\tau_S\) from \((\mathcal{X}(L), \subseteq, \tau_P)\)?

We simply take open upsets of \((\mathcal{X}(L), \subseteq, \tau_P)\)!
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**Lemma:** \(\tau_S\) consists of the open upsets of \((X(L), \subseteq, \tau_P)\).
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Consequently, in the Priestley space \((\mathcal{X}(L), \subseteq, \tau_P)\), \(\tau_P\) is the patch topology of \(\tau_S\) and \(\subseteq\) is the specialization order of \(\tau_S\). Thus we can recover the Priestley space \((\mathcal{X}(L), \subseteq, \tau_P)\) from the space \((\mathcal{X}(L), \tau_S)\).

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**Lemma:** \(\tau_S\) consists of the open upsets of \((\mathcal{X}(L), \subseteq, \tau_P)\).

**Proof:** We already saw that the clopen upsets of \((\mathcal{X}(L), \subseteq, \tau_P)\) are exactly the subsets of \(\mathcal{X}(L)\) of the form \(\phi(a)\) for some \(a \in L\).
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Consequently, in the Priestley space \((\mathcal{X}(L), \subseteq, \tau_P)\), \(\tau_P\) is the patch topology of \(\tau_S\) and \(\subseteq\) is the specialization order of \(\tau_S\). Thus we can recover the Priestley space \((\mathcal{X}(L), \subseteq, \tau_P)\) from the space \((\mathcal{X}^-(L), \tau_S)\).

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Consequently, in the Priestley space \((\mathcal{K}(L), \subseteq, \tau_P)\), \(\tau_P\) is the patch topology of \(\tau_S\) and \(\subseteq\) is the specialization order of \(\tau_S\). Thus we can recover the Priestley space \((\mathcal{K}(L), \subseteq, \tau_P)\) from the space \((\mathcal{K}(L), \tau_S)\).

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We simply take open upsets of \((\mathcal{K}(L), \subseteq, \tau_P)\)!

**Lemma:** \(\tau_S\) consists of the open upsets of \((\mathcal{K}(L), \subseteq, \tau_P)\).

**Proof:** We already saw that the clopen upsets of \((\mathcal{K}(L), \subseteq, \tau_P)\) are exactly the subsets of \(\mathcal{K}(L)\) of the form \(\phi(a)\) for some \(a \in L\). Therefore clopen upsets of \((\mathcal{K}(L), \subseteq, \tau_P)\) are exactly \(S\). Since open upsets of \((\mathcal{K}(L), \subseteq, \tau_P)\) are obtained as the unions of clopen upsets of \((\mathcal{K}(L), \subseteq, \tau_P)\), we conclude that \(\tau_S\) is exactly the open upsets of \((\mathcal{K}(L), \subseteq, \tau_P)\).
Therefore we obtain full balance between \((\mathcal{X}(L), \subseteq, \tau_P)\) and \((\mathcal{X}(L), \tau_S)\).
From Priestley topology to spectral topology

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Given \((\mathcal{X}(L), \subseteq, \tau_P)\), we take the open upsets of \((\mathcal{X}(L), \subseteq, \tau_P)\) to obtain \((\mathcal{X}(L), \tau_S)\);
From Priestley topology to spectral topology

Therefore we obtain full balance between \((\mathcal{X}^0(L), \subseteq, \tau_P)\) and \((\mathcal{X}^0(L), \tau_S)\).

Given \((\mathcal{X}^0(L), \subseteq, \tau_P)\), we take the open upsets of \((\mathcal{X}^0(L), \subseteq, \tau_P)\) to obtain \((\mathcal{X}^0(L), \tau_S)\); and conversely, given \((\mathcal{X}^0(L), \tau_S)\) we take the patch topology of \(\tau_S\) with the specialization order of \(\tau_S\) to obtain \((\mathcal{X}^0(L), \subseteq, \tau_P)\).
Spectral spaces

What we haven’t addressed yet is an abstract topological characterization of those spaces which are homeomorphic to \((\mathcal{K}(L), \tau_S)\) for some bounded distributive lattice \(L\).
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Spectral spaces

What we haven’t addressed yet is an abstract topological characterization of those spaces which are homeomorphic to $\mathcal{K}^*(L, \tau_S)$ for some bounded distributive lattice $L$. We do this now.

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We show that $\tau_S$ is $T_0$. Let $x \neq y$. Then either $x \nsubseteq y$ or $y \nsubseteq x$. Without loss of generality we may assume that $x \nsubseteq y$. Therefore there exists $a \in x - y$. 

Spectral spaces
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We show that \(\tau_S\) is \(T_0\). Let \(x \neq y\). Then either \(x \not\subseteq y\) or \(y \not\subseteq x\). Without loss of generality we may assume that \(x \not\subseteq y\). Therefore there exists \(a \in x - y\). Thus \(x \in \phi(a)\) and \(y \notin \phi(a)\).
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We show that \(\tau_S\) is \(T_0\). Let \(x \neq y\). Then either \(x \not\subseteq y\) or \(y \not\subseteq x\). Without loss of generality we may assume that \(x \not\subseteq y\). Therefore there exists \(a \in x - y\). Thus \(x \in \phi(a)\) and \(y \notin \phi(a)\). This means that there exists a \(\tau_S\)-open set containing \(x\) and missing \(y\). Consequently \(\tau_S\) is \(T_0\).
Spectral spaces

In addition, we have that the clopen upsets of \((X(L), \subseteq, \tau_P)\) are exactly those open subsets of \((X(L), \tau_S)\) which are compact.

Definition: We call a space spectral if it is compact, coherent, and sober.
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In addition, we have that the clopen upsets of $(\mathcal{H}(L), \subseteq, \tau_P)$ are exactly those open subsets of $(\mathcal{H}(L), \tau_S)$ which are compact. The proof of this fact requires some work.
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As a result, we obtain that the family \(E(\mathcal{X}(L), \tau_S)\) of compact open subsets of \((\mathcal{X}(L), \tau_S)\) is a bounded sublattice of \(\tau_S\) which generates the topology \(\tau_S\).

Such spaces are usually called coherent. Thus \((\mathcal{X}(L), \tau_S)\) is \(T_0\), compact, and coherent. In fact, \((\mathcal{X}(L), \tau_S)\) is also a sober space. Because of the lack of time we skip the details. Thus we obtain that \((\mathcal{X}(L), \tau_S)\) is compact, coherent, and sober.

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**Definition:** We call a space *spectral* if it is compact, coherent, and sober.
Spectral duality

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It also shows that there’s a complete balance between bounded distributive lattices and spectral spaces—a result going back to Stone. In particular, this gives us another representation theorem for bounded distributive lattices:
Stone’s representation of bounded distributive lattices: Each bounded distributive lattice is represented as the lattice of compact open subsets of a spectral space.
**Spectral duality**

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Spectral duality

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In particular, this implies the following representation of bounded distributive lattices.

**Topological representation theorem:** Each bounded distributive lattice is isomorphic to a sublattice of $\tau_S$. Therefore each bounded distributive lattice can be represented as a sublattice of the lattice of open subsets of some topological space.
Remark: Note that in fact we have several topological representation theorems for bounded distributive lattices.
Spectral duality

Remark: Note that in fact we have several topological representation theorems for bounded distributive lattices. In Lecture 2 we showed that each bounded distributive lattice $L$ is isomorphic to a sublattice of the lattice of upsets of $(\mathcal{K}(L), \subseteq)$. This in fact is already a topological representation of $L$ because we can view $\mathcal{K}(L)$ as a topological space with the Alexandroff topology $\tau \subseteq$. In Lecture 4 we showed that $L$ is isomorphic to the lattice of clopen upsets of the Priestley dual $L^\ast = (\mathcal{K}(L), \subseteq, \tau_P)$ of $L$. This can be viewed as another topological representation of $L$ since $L$ becomes isomorphic to a sublattice of the lattice of open subsets of $(\mathcal{K}(L), \tau_P)$. In a sense, the topological representation that we obtained in this lecture is the "most economical" because the spectral topology is in fact the intersection of the Alexandroff and the Priestley topologies. That is, $\tau_S = \tau \subseteq \cap \tau_P$. 
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Thus we can develop a duality for distributive lattices by means of either topology and order—Priestley duality—where topology behaves rather nicely;
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Thus we can develop a duality for distributive lattices by means of either topology and order—Priestley duality—where topology behaves rather nicely; or only by means of topology—spectral duality—but then the topology is not as nice as in the other case.
Spectral duality

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Distributive lattices in logic

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Formulae of these calculi are built from propositional variables $p, q, ...$, logical constants $\top$ ("true") and $\bot$ ("false"),
Distributive lattices in logic

We conclude this series of lectures by showing how one can apply the developed theory to obtain various completeness results in logic.

The representation theorems that we obtained in previous lectures readily provide completeness theorems for the following propositional logical systems: Intuitionistic Propositional Calculus (IPC), Classical Propositional Calculus (CPC), and their implication-free fragments.

Formulae of these calculi are built from propositional variables $p$, $q$, ..., logical constants $\top$ (“true”) and $\bot$ (“false”), and logical connectives $\land$ (conjunction), $\lor$ (disjunction), and $\rightarrow$ (implication).
Distributive lattices in logic

One possible description of these systems is based on sequent calculus.
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Recall that a sequent $\Gamma \vdash \Delta$ is an ordered pair where $\Gamma = \varphi_1, ..., \varphi_m$ and $\Delta = \psi_1, ..., \psi_n$ are (possibly empty) finite tuples of formulæ, called contexts.
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Our systems can be axiomatized using the inference rules of the form

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"from sequents $\Gamma_1 \vdash \Delta_1, \ldots, \Gamma_k \vdash \Delta_k$ infer the sequent $\Gamma \vdash \Delta$.”

A proof in each of the systems consists of a succession of sequents each of which is derivable from the previous ones according to the inference rules.
Algebraic semantics

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In the semantics that we will consider, the formulæ will be interpreted by elements of a bounded distributive lattice; those of IPC will be interpreted by elements of a Heyting lattice; and those of CPC—by elements of a Boolean lattice.

Moreover, conjunction will be interpreted by meet, disjunction by join, and implication by the Heyting implication in case of IPC and by the Boolean implication in case of CPC.
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A model of one of our calculi in this semantics thus consists of a bounded distributive lattice $L$ together with a valuation – an assignment to each propositional variable $p$ of an element $v(p) \in L$. 
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The valuation is then extended to all formulæ by induction:

\[
v(\top) = 1,
\]

\[
v(\bot) = 0,
\]

\[
v(\phi \land \psi) = v(\phi) \land v(\psi),
\]

\[
v(\phi \lor \psi) = v(\phi) \lor v(\psi),
\]

and for IPC (resp. CPC), \( L \) must be a Heyting lattice (resp. Boolean lattice), and additionally

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v(\phi \rightarrow \psi) = v(\phi) \rightarrow v(\psi).
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A sequent $\varphi_1, \ldots, \varphi_m \vdash \psi_1, \ldots, \psi_n$ is said to be true in such a model if

$$v(\varphi_1) \land \cdots \land v(\varphi_m) \leq v(\psi_1) \lor \cdots \lor v(\psi_n)$$

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A calculus is said to be sound with respect to this semantics if any sequent which is derivable starting “from nothing”, i. e. starting from an empty succession of sequents, is true in all models of this semantics.
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A sequent \( \varphi_1, \ldots, \varphi_m \vdash \psi_1, \ldots, \psi_n \) is said to be true in such a model if

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In principle the only thing we need to know about the inference rules is that they ensure soundness of the corresponding system with respect to the semantics.
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As a simple example, the inference rule

\[ \varphi \vdash \varphi \]

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As a more complicated example, consider the cut rule

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This rule corresponds to the fact that in any distributive lattice, if

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A calculus is said to be complete with respect to a class of models in this semantics if any sequent which is true in all models from that class is derivable in the above sense.
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The formulæ \( \varphi \) and \( \psi \) are called provably equivalent if the sequents \( \varphi \vdash \psi \) and \( \psi \vdash \varphi \) are both derivable in the calculus.
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Identifying provably equivalent formulæ one obtains a lattice of appropriate type equipped with the valuation \( v \) which assigns to a formula \( \varphi \) its equivalence class.
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Identifying provably equivalent formulæ one obtains a lattice of appropriate type equipped with the valuation $\nu$ which assigns to a formula $\varphi$ its equivalence class.

In this way, we obtain a model, and it is then not difficult to see that a sequent is derivable iff it is true in this model.
Completeness

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Our first representation theorem of Lecture 2 implies that each bounded distributive lattice \( L \) is isomorphic to a sublattice of the lattice \( \mathcal{U}(P) \) of upsets of some poset \( P \).
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Our first representation theorem of Lecture 2 implies that each bounded distributive lattice $L$ is isomorphic to a sublattice of the lattice $\mathcal{U}(P)$ of upsets of some poset $P$.

This theorem implies that the implication-free fragment of IPC is complete with respect to the relational semantics—the semantics in which the only models allowed are those in which formulæ are interpreted as upsets of a poset $P$, the conjunction as set-theoretic intersection, and the disjunction as set-theoretic union.
Relational completeness

Similarly, the representation of Heyting lattices provides us with the following completeness of IPC:

\[
\text{IPC is complete with respect to the class of all posets.}
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For those familiar with Kripke semantics of IPC, the above completeness is just a reformulation of the Kripke completeness of IPC. Put differently, Kripke completeness of IPC is nothing more but a representation of Heyting lattices as lattices of upsets of posets!
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**Relational completeness of IPC:** If we interpret formulæ of IPC as upsets, $\land$ as set-theoretic intersection, $\lor$ as set-theoretic union, and $\phi \rightarrow \psi$ as

$$P - \downarrow (\nu(\varphi) - \nu(\psi))$$

$$= \{ w \in P : \text{for all } w' \succeq w, \text{if } w' \in \nu(\varphi), \text{then } w' \in \nu(\psi) \} ,$$
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Completeness for CPC

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In the case of Boolean lattices the order \( \leq \) of the poset \( P \) becomes trivial. Thus we arrive at the following well-known completeness of CPC:

**Completeness of CPC:** If we interpret formulæ of CPC as subsets of a set, \( \land \) as set-theoretic intersection, \( \lor \) as set-theoretic union, and \( \phi \rightarrow \psi \) as \( (S - \nu(\varphi)) \cup \nu(\psi) \), then CPC is complete with respect to the class of all sets.
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On the other hand, IPC is complete with respect to an infinite class of finite models—another famous result from the thirties by Stanislaw Jaśkowski.
Topological completeness

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**Topological completeness of IPC:** If we interpret formulæ of IPC as open subsets, $\land$ as set-theoretic intersection, $\lor$ as set-theoretic union, and $\phi \rightarrow \psi$ as

$$X - \nu(\varphi) - \nu(\psi) = \text{int}((X - \nu(\varphi)) \cup \nu(\psi)),$$

then IPC is complete with respect to the class of all topological spaces.
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This restriction is again **not** possible in the case of IPC.
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- We have developed basics of lattice theory.
- We have characterized distributive lattices as those lattices which do not have the diamond and pentagon configurations.
- We have introduced Boolean lattices and Heyting lattices, which form important subclasses of the class of distributive lattices.
- We have developed the Birkhoff duality between finite distributive lattices and finite posets.
Summary

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- We have obtained the Stone duality for Boolean lattices and the Esakia duality for Heyting lattices from the Priestley duality.

We have also developed the spectral duality which is an alternative of the Priestley duality. As a result we have obtained relational and topological representations of bounded distributive lattices, Heyting lattices, and Boolean lattices. We have given applications of these representation theorems to logic. In particular, we have discussed several relational and topological completeness theorems for the intuitionistic and classical logics, and their implication-free fragments.
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We have given applications of these representation theorems to logic. In particular, we have discussed several relational and topological completeness theorems for the intuitionistic and classical logics, and their implication-free fragments.
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