Lattices and Topology

Guram Bezhanishvili and Mamuka Jibladze

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Lecture 4: Duality

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- We defined subspaces, continuous maps, and homeomorphisms.
- We also introduced T_0, T_1, Hausdorff, and sober spaces.
- We described the specialization order of a T_0-space and discussed the complete balance between Alexandroff T_0-spaces and posets. In particular, we discussed the complete balance between finite T_0-spaces and finite posets.

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- We also showed that a subset of a compact Hausdorff space is compact iff it is closed.
- In addition, we showed that each 1-1 onto continuous map between compact Hausdorff spaces is a homeomorphism.
- We concluded the lecture by defining Stone spaces and giving some nontrivial examples of Stone spaces.

Short outline of lecture 4

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- Priestley duality for distributive lattices

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In order to single out this sublattice we developed all the necessary background from topology in Lecture 3.

The main goal of this lecture is to take advantage of the topological machinery which will allow us to characterize the needed sublattice by the hybrid of order-theoretic and topological methods.

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Consider the following collection

$$\mathscr{B} = \{\phi(a) - \phi(b) : a, b \in L\}$$

of subsets of $\mathscr{X}(L)$.

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We give an abstract characterization of such spaces.

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Further reduction is possible thanks to the Alexander subbasis lemma which states that if the topology is generated by the unions of finite intersections of a given family $\mathcal{S}$, then in order to verify compactness of the space, it is sufficient to verify that each cover of the space by elements of $\mathcal{S}$ has a finite subcover.

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Let $\mathscr{X}(L) = \bigcup\{\phi(a_\sigma) : \sigma \in \Sigma\} \cup \bigcup\{\mathscr{X}(L) - \phi(b_\delta) : \delta \in \Delta\}$.

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If $F \cap I = \emptyset$, then, by Stone's lemma, there exists $x \in \mathscr{X}(L)$ such that $F \subseteq x$ and $x \cap I = \emptyset$. Therefore $x \in \phi(b_\delta)$ for each $\delta \in \Delta$ and $x \notin \phi(a_\sigma)$ for each $\sigma \in \Sigma$.

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$x \notin \bigcup\{\phi(a_\sigma) : \sigma \in \Sigma\} \cup \bigcup\{\mathscr{X}(L) - \phi(b_\delta) : \delta \in \Delta\}$, which means that $\bigcup\{\phi(a_\sigma) : \sigma \in \Sigma\} \cup \bigcup\{\mathscr{X}(L) - \phi(b_\delta) : \delta \in \Delta\}$ is not a cover of $\mathscr{X}(L)$.

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The obtained contradiction proves that $F \cap I \neq \emptyset$.

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But then $\phi(b_{\delta_1}) \cap \cdots \cap \phi(b_{\delta_n}) \subseteq \phi(c)$ and
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Thus $(\mathscr{X}(L), \tau_P)$ is compact.

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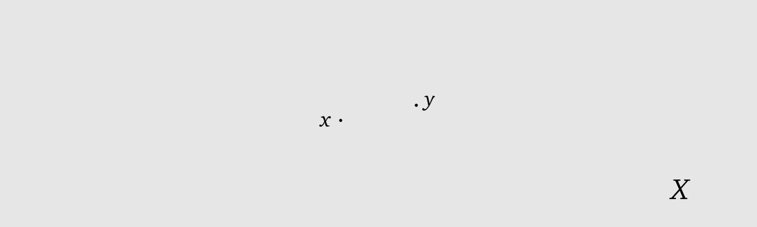

For any points $x, y \in X$,

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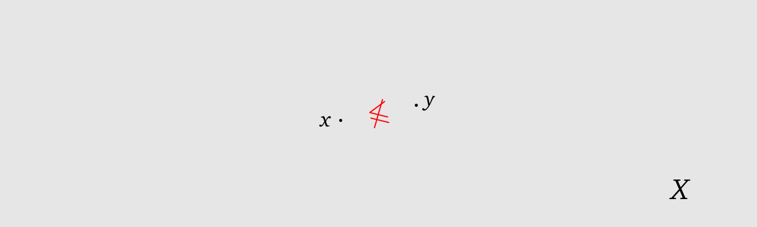

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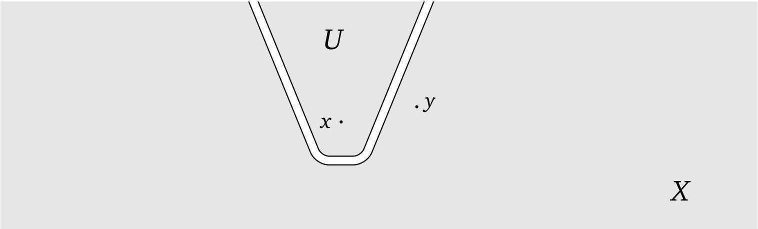

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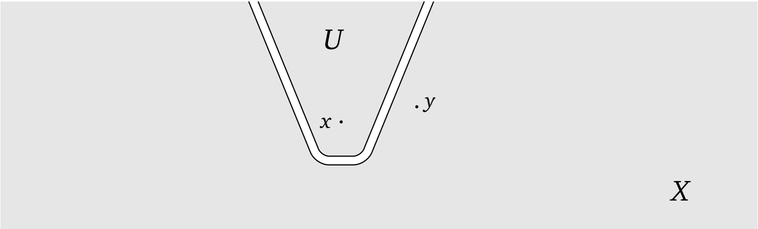

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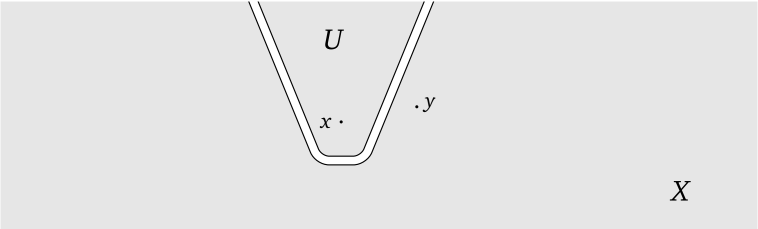

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Thus, every bounded distributive lattice L gives rise to the Priestley space L_*.

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Conversely, for each Priestley space $(X, \leq, \tau)$, let X^* be the set of clopen upsets of X.

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Lemma: $(X^*, \cup, \cap, \emptyset, X)$ forms a bounded distributive lattice.

Proof: Clearly $\emptyset$ and X are clopen upsets, and the union and intersection of two clopens is again clopen. Since $\cup$ and $\cap$ distribute over each other, it follows that $(X^*, \cup, \cap, \emptyset, X)$ forms a bounded distributive lattice.

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Since U is an upset, for each $y \notin U$ we have $x \not\subseteq y$. Therefore there exist a_y such that $a_y \in x$ and $a_y \notin y$. Thus $x \in \phi(a_y)$ and $y \notin \phi(a_y)$. This means that $x \notin \mathscr{X}(L) - \phi(a_y)$ and $y \in \mathscr{X}(L) - \phi(a_y)$.

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Conversely, we can view each Stone space (X, τ) as the Priestley space $(X, \leq, \tau)$ with the discrete $\leq$.

The Stone duality for Boolean lattices

Now suppose that L is a Boolean lattice. Then it is easy to show that the poset of prime filters of L is discrete.

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Conversely, we can view each Stone space (X, τ) as the Priestley space $(X, \leq, \tau)$ with the discrete $\leq$. Then X^* becomes simply the lattice of clopen subsets of X, which is clearly a Boolean lattice because it is closed under set-theoretic complement.

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In particular, we obtain the following representation theorem for Boolean lattices.

Stone's representation of Boolean lattices: Each Boolean lattice can be represented as the lattice of clopen subsets of a Stone space.

The Esakia duality for Heyting lattices

Let L be a Heyting lattice and let $(\mathscr{X}(L), \subseteq, \tau_P)$ be the Priestley space of L. Then we have

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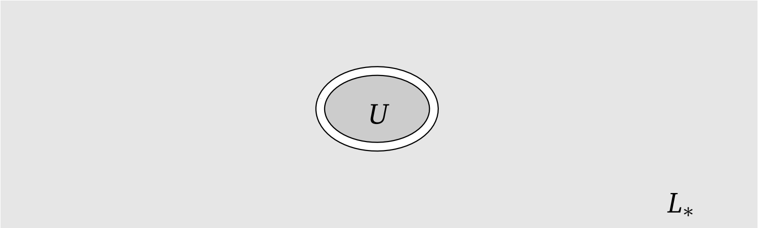
Esakia spaces



Theorem: If L is a Heyting lattice, then for each clopen U of L_*

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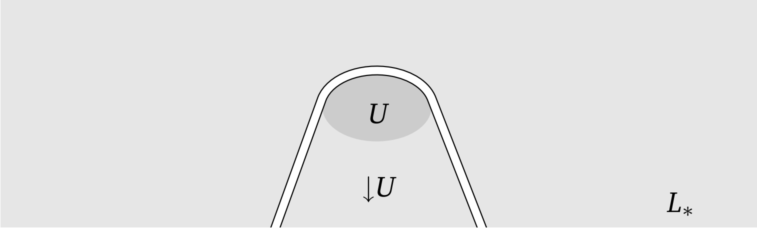

Theorem: If L is a Heyting lattice, then for each clopen U of L_* the downset $\downarrow U$ is also clopen.

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Thus $\downarrow U$ is clopen.

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We call such spaces Esakia spaces.

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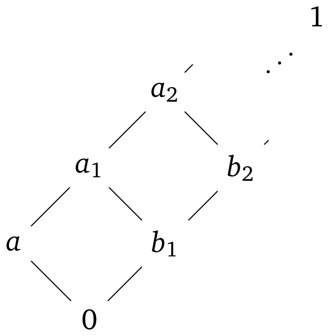

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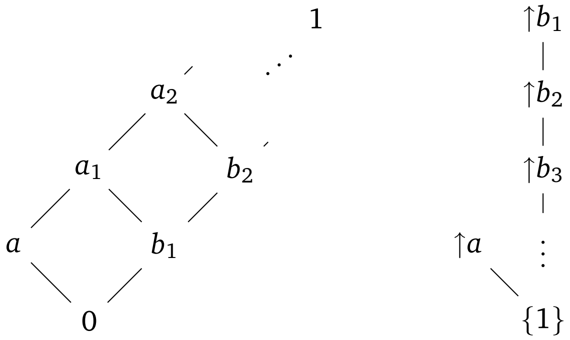

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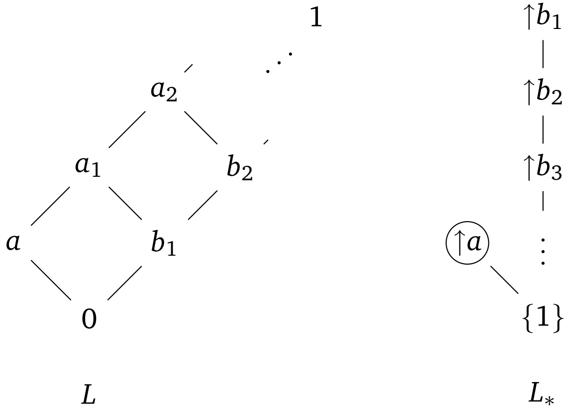

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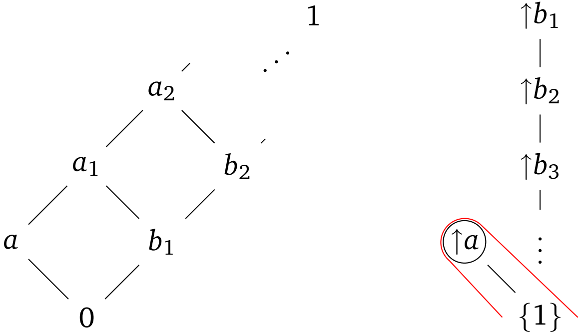

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Esakia's representation of Heyting lattices: Each Heyting lattice can be represented as the lattice of clopen upsets of an Esakia space.

Three dualities

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Boolean lattices	$\subset$	Heyting lattices	$\subset$	Distributive lattices
$\updownarrow$		$\updownarrow$		$\updownarrow$
Stone spaces	$\subset$	Esakia spaces	$\subset$	Priestley spaces