Lattices and Topology

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Lecture 4: Duality
The previous lecture was dedicated to topology. We defined topological spaces; introduced the concepts of open and closed sets, and of the interior and closure of a set. We defined subspaces, continuous maps, and homeomorphisms. We also introduced $T_0$, $T_1$, Hausdorff, and sober spaces. We described the specialization order of a $T_0$-space and discussed the complete balance between Alexandroff $T_0$-spaces and posets. In particular, we discussed the complete balance between finite $T_0$-spaces and finite posets.
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- In addition, we showed that each 1-1 onto continuous map between compact Hausdorff spaces is a homeomorphism.
- We concluded the lecture by defining **Stone spaces** and giving some nontrivial examples of Stone spaces.
Lecture 4: Duality

Short outline of lecture 4

Priestley duality for distributive lattices

Stone duality for Boolean lattices

Esakia duality for Heyting lattices
Lecture 4: Duality

- Priestley duality for distributive lattices
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In order to single out this sublattice we developed all the necessary background from topology in Lecture 3.

The main goal of this lecture is to take advantage of the topological machinery which will allow us to characterize the needed sublattice by the hybrid of order-theoretic and topological methods.
Priestley topology

Let $L$ be a bounded distributive lattice.
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$$\phi(a) = \{x \in \mathcal{X}(L) : a \in x\}.$$
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\section*{Priestley topology}
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**Lemma:** If $U, V \in \mathcal{B}$, then $U \cap V \in \mathcal{B}$. 


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**Lemma:** If $U, V \in \mathcal{B}$, then $U \cap V \in \mathcal{B}$.

**Proof:** Let $U, V \in \mathcal{B}$. Then there exist $a, b, c, d \in L$ such that $U = \phi(a) - \phi(b)$ and $V = \phi(c) - \phi(d)$. 
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= \phi(a \land c) \cap (\mathcal{K}(L) - \phi(b \lor d))
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On the other hand, $\mathcal{B}$ is not closed under arbitrary unions in general.
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We give an abstract characterization of such spaces.
Theorem: The Priestley topology is compact.
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**Theorem:** The Priestley topology is compact.

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Further reduction is possible thanks to the Alexander subbasis lemma.
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Further reduction is possible thanks to the Alexander subbasis lemma which states that if the topology is generated by the unions of finite intersections of a given family $S$, then in order to verify compactness of the space, it is sufficient to verify that each cover of the space by elements of $S$ has a finite subcover.
Priestley topology

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\{ \phi(a_\sigma) : \sigma \in \Sigma \} \cup \{ \mathcal{X}(L) - \phi(b_\delta) : \delta \in \Delta \}.
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Let $\mathcal{X}(L) = \bigcup\{\phi(a_\sigma) : \sigma \in \Sigma\} \cup \bigcup\{\mathcal{X}(L) - \phi(b_\delta) : \delta \in \Delta\}$. 

Priestley topology
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We let $F$ be the smallest filter containing $\{b_\delta : \delta \in \Delta\}$ and $I$ be the smallest ideal containing $\{a_\sigma : \sigma \in \Sigma\}$. Then, by Stone's lemma, there exists $x \in X(L)$ such that $F \subseteq x$ and $x \cap I = \emptyset$. Therefore $x \in \varphi(b_\delta)$ for each $\delta \in \Delta$ and $x \notin \varphi(a_\sigma)$ for each $\sigma \in \Sigma$. Thus $\bigcup \{\varphi(a_\sigma) : \sigma \in \Sigma\} \cup \bigcup \{X(L) - \varphi(b_\delta) : \delta \in \Delta\}$ is not a cover of $X(L)$. The obtained contradiction proves that $F \cap I \neq \emptyset$. 
We let $F$ be the smallest filter containing ${b_\delta : \delta \in \Delta}$ and $I$ be the smallest ideal containing ${a_\sigma : \sigma \in \Sigma}$.

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If $F \cap I = \emptyset$, then, by Stone’s lemma, there exists $x \in \mathcal{X}(L)$ such that $F \subseteq x$ and $x \cap I = \emptyset$. Therefore $x \in \phi(b_\delta)$ for each $\delta \in \Delta$ and $x \notin \phi(a_\sigma)$ for each $\sigma \in \Sigma$. 
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$x \notin \bigcup\{\phi(a_\sigma) : \sigma \in \Sigma\} \cup \bigcup\{\mathcal{X}(L) - \phi(b_\delta) : \delta \in \Delta\}$, which means that $\bigcup\{\phi(a_\sigma) : \sigma \in \Sigma\} \cup \bigcup\{\mathcal{X}(L) - \phi(b_\delta) : \delta \in \Delta\}$ is not a cover of $\mathcal{X}(L)$. 

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The obtained contradiction proves that $F \cap I \neq \emptyset$. 
Since $F \cap I \neq \emptyset$, there exists $c \in L$ such that $b_{\delta_1} \land \cdots \land b_{\delta_n} \leq c$ for some $\delta_1, \ldots, \delta_n \in \Delta$
Priestley topology

Since \( F \cap I \neq \emptyset \), there exists \( c \in L \) such that \( b_{\delta_1} \land \cdots \land b_{\delta_n} \leq c \) for some \( \delta_1, \ldots, \delta_n \in \Delta \) and \( c \leq a_{\sigma_1} \lor \cdots \lor a_{\sigma_k} \) for some \( \sigma_1, \ldots, \sigma_k \in \Sigma \).
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But then $\phi(b_{\delta_1}) \cap \cdots \cap \phi(b_{\delta_n}) \subseteq \phi(c)$ and $\phi(c) \subseteq \phi(a_{\sigma_1}) \cup \cdots \cup \phi(a_{\sigma_k})$. 

Therefore $\phi(b_{\delta_1}) \cap \cdots \cap \phi(b_{\delta_n}) \subseteq \phi(a_{\sigma_1}) \cup \cdots \cup \phi(a_{\sigma_k})$. 

Thus $\phi(a_{\sigma_1}) \cup \cdots \cup \phi(a_{\sigma_k}) = X(L) - \phi(b_{\delta_1}) \cup \cdots \cup X(L) - \phi(b_{\delta_k})$, which implies that there is a finite subcover of $X(L)$.
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Therefore $\phi(b_{\delta_1}) \cap \cdots \cap \phi(b_{\delta_n}) \subseteq \phi(a_{\sigma_1}) \cup \cdots \cup \phi(a_{\sigma_k})$. Thus

$\phi(a_{\sigma_1}) \cup \cdots \cup \phi(a_{\sigma_k}) \cup (\mathcal{X}'(L) - \phi(b_{\delta_1})) \cup \cdots \cup (\mathcal{X}'(L) - \phi(b_{\delta_k})) = \mathcal{X}'(L)$,
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Therefore $\phi(b_{\delta_1}) \cap \cdots \cap \phi(b_{\delta_n}) \subseteq \phi(a_{\sigma_1}) \cup \cdots \cup \phi(a_{\sigma_k})$. Thus $\phi(a_{\sigma_1}) \cup \cdots \cup \phi(a_{\sigma_k}) \cup (\mathcal{X}(L) - \phi(b_{\delta_1})) \cup \cdots \cup (\mathcal{X}(L) - \phi(b_{\delta_k})) = \mathcal{X}(L)$, which implies that there is a finite subcover of $\mathcal{X}(L)$. 
Since $F \cap I \neq \emptyset$, there exists $c \in L$ such that $b_{\delta_1} \land \cdots \land b_{\delta_n} \leq c$ for some $\delta_1, \ldots, \delta_n \in \Delta$ and $c \leq a_{\sigma_1} \lor \cdots \lor a_{\sigma_k}$ for some $\sigma_1, \ldots, \sigma_k \in \Sigma$.

But then $\phi(b_{\delta_1}) \cap \cdots \cap \phi(b_{\delta_n}) \subseteq \phi(c)$ and $\phi(c) \subseteq \phi(a_{\sigma_1}) \cup \cdots \cup \phi(a_{\sigma_k})$.

Therefore $\phi(b_{\delta_1}) \cap \cdots \cap \phi(b_{\delta_n}) \subseteq \phi(a_{\sigma_1}) \cup \cdots \cup \phi(a_{\sigma_k})$. Thus $\phi(a_{\sigma_1}) \cup \cdots \cup \phi(a_{\sigma_k}) \cup (\mathcal{X}(L) - \phi(b_{\delta_1})) \cup \cdots \cup (\mathcal{X}(L) - \phi(b_{\delta_k})) = \mathcal{X}(L)$, which implies that there is a finite subcover of $\mathcal{X}(L)$. Thus $(\mathcal{X}(L), \tau_P)$ is compact.
Priestley spaces

A Priestley space is a triple $(X, \leq, \tau)$ such that $(X, \leq)$ is a poset, $(X, \tau)$ is a compact space, and the order and topology are connected by the Priestley separation axiom:
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Priestley spaces

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\[
\begin{array}{c}
x \cdot \not\leq \cdot y \\
X
\end{array}
\]

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A **Priestley space** is a triple \((X, \leq, \tau)\) such that \((X, \leq)\) is a poset, \((X, \tau)\) is a compact space, and the order and topology are connected by the **Priestley separation axiom**: 

For any points \(x, y \in X\), if \(x \not\leq y\), then there exists a clopen upset \(U\) of \(X\) such that \(x \in U\) and \(y \notin U\).
Priestley spaces

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This, in particular, implies that each Priestley space is Hausdorff and zero-dimensional.
Priestley spaces

A Priestley space is a triple \((X, \leq, \tau)\) such that \((X, \leq)\) is a poset, \((X, \tau)\) is a compact space, and the order and topology are connected by the Priestley separation axiom:

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This, in particular, implies that each Priestley space is Hausdorff and zero-dimensional hence a Stone space.
Priestley duality

For a bounded distributive lattice $L$, let $L_* = (\mathcal{H}(L), \subseteq, \tau_P)$.
Priestley duality

For a bounded distributive lattice $L$, let $L_\ast = (\mathcal{X}(L), \subseteq, \tau_P)$.

**Theorem:** If $L$ is a bounded distributive lattice, then $L_\ast$ is a Priestley space.
Priestley duality

For a bounded distributive lattice $L$, let $L_* = (\mathcal{X}(L), \subseteq, \tau_P)$.

**Theorem:** If $L$ is a bounded distributive lattice, then $L_*$ is a Priestley space.

**Proof:** It is obvious that $(\mathcal{X}(L), \subseteq)$ is a poset.
Priestley duality

For a bounded distributive lattice $L$, let $L_* = \langle \mathcal{K}(L), \subseteq, \tau_P \rangle$.

**Theorem:** If $L$ is a bounded distributive lattice, then $L_*$ is a Priestley space.

**Proof:** It is obvious that $(\mathcal{K}(L), \subseteq)$ is a poset. We already showed that $(\mathcal{K}(L), \tau_P)$ is a compact space.
Priestley duality

For a bounded distributive lattice $L$, let $L_* = (\mathcal{X}(L), \subseteq, \tau_P)$.

**Theorem:** If $L$ is a bounded distributive lattice, then $L_*$ is a Priestley space.

**Proof:** It is obvious that $(\mathcal{X}(L), \subseteq)$ is a poset. We already showed that $(\mathcal{X}(L), \tau_P)$ is a compact space. It is left to verify that $L_*$ satisfies the Priestley separation axiom.
Priestley duality

For a bounded distributive lattice $L$, let $L_* = (\mathcal{X}(L), \subseteq, \tau_P)$.

**Theorem:** If $L$ is a bounded distributive lattice, then $L_*$ is a Priestley space.

**Proof:** It is obvious that $(\mathcal{X}(L), \subseteq)$ is a poset. We already showed that $(\mathcal{X}(L), \tau_P)$ is a compact space. It is left to verify that $L_*$ satisfies the Priestley separation axiom. Let $x \not\subseteq y$. Then there exists $a \in x \setminus y$. Therefore $x \in \phi(a)$ and $y \not\in \phi(a)$. We already verified that $\phi(a)$ is an upset. Moreover both $\phi(a)$ and $\mathcal{X}(L) \setminus \phi(a)$ belong to $\mathcal{B}$. Therefore $\phi(a)$ is clopen. Consequently $L_*$ is a Priestley space. Thus, every bounded distributive lattice $L$ gives rise to the Priestley space $L_*$. 
Priestley duality

For a bounded distributive lattice $L$, let $L_* = (\mathcal{X}(L), \subseteq, \tau_P)$.

**Theorem:** If $L$ is a bounded distributive lattice, then $L_*$ is a Priestley space.

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For a bounded distributive lattice $L$, let $L_* = (\mathcal{X}(L), \subseteq, \tau_P)$.

**Theorem:** If $L$ is a bounded distributive lattice, then $L_*$ is a Priestley space.

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Priestley duality

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Priestley duality

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**Theorem:** If $L$ is a bounded distributive lattice, then $L_*$ is a Priestley space.

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Priestley duality

For a bounded distributive lattice $L$, let $L_* = (\mathcal{X}(L), \subseteq, \tau_P)$.

**Theorem:** If $L$ is a bounded distributive lattice, then $L_*$ is a Priestley space.

**Proof:** It is obvious that $(\mathcal{X}(L), \subseteq)$ is a poset. We already showed that $(\mathcal{X}(L), \tau_P)$ is a compact space. It is left to verify that $L_*$ satisfies the Priestley separation axiom. Let $x \not\subseteq y$. Then there exists $a \in x - y$. Therefore $x \in \phi(a)$ and $y \not\in \phi(a)$. We already verified that $\phi(a)$ is an upset. Moreover both $\phi(a)$ and $\mathcal{X}(L) - \phi(a)$ belong to $\mathcal{B}$. Therefore $\phi(a)$ is clopen. Consequently $L_*$ is a Priestley space.
Priestley duality

For a bounded distributive lattice $L$, let $L_* = (\mathcal{E}(L), \subseteq, \tau_P)$.

**Theorem:** If $L$ is a bounded distributive lattice, then $L_*$ is a Priestley space.

**Proof:** It is obvious that $(\mathcal{E}(L), \subseteq)$ is a poset. We already showed that $(\mathcal{E}(L), \tau_P)$ is a compact space. It is left to verify that $L_*$ satisfies the Priestley separation axiom. Let $x \not\subseteq y$. Then there exists $a \in x - y$. Therefore $x \in \phi(a)$ and $y \not\in \phi(a)$. We already verified that $\phi(a)$ is an upset. Moreover both $\phi(a)$ and $\mathcal{E}(L) - \phi(a)$ belong to $\mathcal{B}$. Therefore $\phi(a)$ is clopen. Consequently $L_*$ is a Priestley space.

Thus, every bounded distributive lattice $L$ gives rise to the Priestley space $L_*$. 
Priestley duality

Conversely, for each Priestley space \((X, \leq, \tau)\), let \(X^*\) be the set of clopen upsets of \(X\).
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**Lemma:** \((X^*, \cup, \cap, \emptyset, X)\) forms a bounded distributive lattice.
Conversely, for each Priestley space \((X, \leq, \tau)\), let \(X^*\) be the set of clopen upsets of \(X\).

**Lemma:** \((X^*, \cup, \cap, \emptyset, X)\) forms a bounded distributive lattice.

**Proof:** Clearly \(\emptyset\) and \(X\) are clopen upsets, and the union and intersection of two clopens is again clopen.
Conversely, for each Priestley space $\langle X, \leq, \tau \rangle$, let $X^*$ be the set of clopen upsets of $X$.

**Lemma:** $\langle X^*, \cup, \cap, \emptyset, X \rangle$ forms a bounded distributive lattice.

**Proof:** Clearly $\emptyset$ and $X$ are clopen upsets, and the union and intersection of two clopens is again clopen. Since $\cup$ and $\cap$ distribute over each other, it follows that $\langle X^*, \cup, \cap, \emptyset, X \rangle$ forms a bounded distributive lattice.
Priestley duality

Thus, we have the following correspondences:

\[ L \mapsto L^* \mapsto L^{**} \quad \text{and} \quad X \mapsto X^* \mapsto X^{**} \]

Lemma: \( \phi: L \to L^{**} \) is an isomorphism.

Proof: We already saw that \( \phi \) is a 1-1 bounded lattice homomorphism. It is left to be shown that it is onto.

Let \( U \) be a clopen upset of \( L^* \) and \( x \in U \).

We show that there exists \( a \in L \) such that \( x \in \phi(a) \subseteq U \).
Priestley duality

Thus, we have the following correspondences:

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**Lemma:** \( \varphi : L \to L^*\) is an isomorphism.

**Proof:** We already saw that \( \varphi \) is a 1-1 bounded lattice homomorphism. It is left to be shown that it is onto.

Let \( U \) be a clopen upset of \( L^* \) and \( x \in U \).
Thus, we have the following correspondences:

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**Lemma:** \( \phi : L \rightarrow L^{**} \) is an isomorphism.

**Proof:** We already saw that \( \phi \) is a 1-1 bounded lattice homomorphism. It is left to be shown that it is onto.

Let \( U \) be a clopen upset of \( L^* \) and \( x \in U \). We show that there exists \( a \in L \) such that \( x \in \phi(a) \subseteq U \).
Since $U$ is an upset, for each $y \notin U$ we have $x \subsetneq y$. 
Since $U$ is an upset, for each $y \notin U$ we have $x \nsubseteq y$. Therefore there exist $a_y$ such that $a_y \in x$ and $a_y \notin y$. 

This means that $x \notin \phi(a_y)$ and $y \notin \phi(a_y)$. 

Therefore $\{X(L) - \phi(a_y) : y \notin U\}$ is a cover of $X(L) - U$. Since $U$ is clopen, so is $X(L) - U$. Thus it is compact. Therefore there exists a finite subcover of $X(L) - U$. This means that there exist $y_1, \ldots, y_n \in X(L) - U$ such that $X(L) - U \subseteq (X(L) - \phi(a_{y_1})) \cup \cdots \cup (X(L) - \phi(a_{y_n}))$. 

Consequently $x \in \phi(a_{y_1}) \cap \cdots \cap \phi(a_{y_n}) \subseteq U$. This implies $x \in \phi(a_{y_1} \wedge \cdots \wedge a_{y_n}) \subseteq U$. 

Priestley duality
Since $U$ is an upset, for each $y \notin U$ we have $x \nsubseteq y$. Therefore there exist $a_y$ such that $a_y \in x$ and $a_y \notin y$. Thus $x \in \phi(a_y)$ and $y \notin \phi(a_y)$. 

{X(L) - \phi(a_y) : y \notin U}$ is a cover of $X(L) - U$. Since $U$ is clopen, so is $X(L) - U$. Thus it is compact. Therefore there exists a finite subcover of $X(L) - U$. This means that there exist $y_1,...,y_n \in X(L) - U$ such that $X(L) - U \subseteq (X(L) - \phi(a_{y_1})) \cup \cdots \cup (X(L) - \phi(a_{y_n}))$. Consequently $x \in \phi(a_{y_1}) \cap \cdots \cap \phi(a_{y_n}) \subseteq U$. This implies $x \in \phi(a_{y_1} \land \cdots \land a_{y_n}) \subseteq U$. 

Priestley duality
Since $U$ is an upset, for each $y \notin U$ we have $x \not\subseteq y$. Therefore there exist $a_y$ such that $a_y \in x$ and $a_y \notin y$. Thus $x \in \phi(a_y)$ and $y \notin \phi(a_y)$. This means that $x \notin \mathcal{X}(L) - \phi(a_y)$ and $y \in \mathcal{X}(L) - \phi(a_y)$.
Since $U$ is an upset, for each $y \notin U$ we have $x \not\subseteq y$. Therefore there exist $a_y$ such that $a_y \in x$ and $a_y \not\in y$. Thus $x \in \phi(a_y)$ and $y \not\in \phi(a_y)$. This means that $x \notin \mathcal{X}(L) - \phi(a_y)$ and $y \in \mathcal{X}(L) - \phi(a_y)$. Therefore $\{ \mathcal{X}(L) - \phi(a_y) : y \notin U \}$ is a cover of $\mathcal{X}(L) - U$. 


Since $U$ is an upset, for each $y \notin U$ we have $x \nsubseteq y$. Therefore there exist $a_y$ such that $a_y \in x$ and $a_y \notin y$. Thus $x \in \phi(a_y)$ and $y \notin \phi(a_y)$. This means that $x \notin \mathcal{X}(L) - \phi(a_y)$ and $y \in \mathcal{X}(L) - \phi(a_y)$. Therefore $\{ \mathcal{X}(L) - \phi(a_y) : y \notin U \}$ is a cover of $\mathcal{X}(L) - U$.

Since $U$ is clopen, so is $\mathcal{X}(L) - U$. 

\textbf{Priestley duality}
Since $U$ is an upset, for each $y \notin U$ we have $x \not\subseteq y$. Therefore there exist $a_y$ such that $a_y \in x$ and $a_y \not\in y$. Thus $x \in \phi(a_y)$ and $y \not\in \phi(a_y)$. This means that $x \not\in \mathcal{X}^-(L) - \phi(a_y)$ and $y \in \mathcal{X}^-(L) - \phi(a_y)$. Therefore $\{\mathcal{X}^-(L) - \phi(a_y) : y \notin U\}$ is a cover of $\mathcal{X}^-(L) - U$.

Since $U$ is clopen, so is $\mathcal{X}^-(L) - U$. Thus it is compact.
Priestley duality

Since $U$ is an upset, for each $y \notin U$ we have $x \not\subseteq y$. Therefore there exist $a_y$ such that $a_y \in x$ and $a_y \notin y$. Thus $x \in \phi(a_y)$ and $y \notin \phi(a_y)$. This means that $x \notin \mathcal{X}(L) - \phi(a_y)$ and $y \in \mathcal{X}(L) - \phi(a_y)$. Therefore $\{ \mathcal{X}(L) - \phi(a_y) : y \notin U \}$ is a cover of $\mathcal{X}(L) - U$.

Since $U$ is clopen, so is $\mathcal{X}(L) - U$. Thus it is compact. Therefore there exists a finite subcover of $\mathcal{X}(L) - U$. 
Priestley duality

Since \( U \) is an upset, for each \( y \notin U \) we have \( x \not\subseteq y \). Therefore there exist \( a_y \) such that \( a_y \in x \) and \( a_y \notin y \). Thus \( x \in \phi(a_y) \) and \( y \notin \phi(a_y) \). This means that \( x \notin \mathcal{K}(L) - \phi(a_y) \) and \( y \in \mathcal{K}(L) - \phi(a_y) \). Therefore \( \{ \mathcal{K}(L) - \phi(a_y) : y \notin U \} \) is a cover of \( \mathcal{K}(L) - U \).

Since \( U \) is clopen, so is \( \mathcal{K}(L) - U \). Thus it is compact. Therefore there exists a finite subcover of \( \mathcal{K}(L) - U \). This means that there exist \( y_1, \ldots, y_n \in \mathcal{K}(L) - U \) such that 
\[
\mathcal{K}(L) - U \subseteq (\mathcal{K}(L) - \phi(a_{y_1})) \cup \cdots \cup (\mathcal{K}(L) - \phi(a_{y_n})).
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Since \( U \) is an upset, for each \( y \notin U \) we have \( x \not\subseteq y \). Therefore there exist \( a_y \) such that \( a_y \in x \) and \( a_y \notin y \). Thus \( x \in \phi(a_y) \) and \( y \notin \phi(a_y) \). This means that \( x \notin \mathcal{X}^-(L) - \phi(a_y) \) and \( y \in \mathcal{X}^+(L) - \phi(a_y) \). Therefore \( \{ \mathcal{X}^-(L) - \phi(a) : y \notin U \} \) is a cover of \( \mathcal{X}^+(L) - U \).

Since \( U \) is clopen, so is \( \mathcal{X}^+(L) - U \). Thus it is compact. Therefore there exists a finite subcover of \( \mathcal{X}^+(L) - U \). This means that there exist \( y_1, \ldots, y_n \in \mathcal{X}^+(L) - U \) such that
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\mathcal{X}^+(L) - U \subseteq (\mathcal{X}^+(L) - \phi(a_{y_1})) \cup \cdots \cup (\mathcal{X}^+(L) - \phi(a_{y_n})).
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Consequently \( x \in \phi(a_{y_1}) \cap \cdots \cap \phi(a_{y_n}) \subseteq U \).
Since $U$ is an upset, for each $y \notin U$ we have $x \notin y$. Therefore there exist $a_y$ such that $a_y \in x$ and $a_y \notin y$. Thus $x \in \phi(a_y)$ and $y \notin \phi(a_y)$. This means that $x \notin \mathcal{X}(L) - \phi(a_y)$ and $y \in \mathcal{X}(L) - \phi(a_y)$. Therefore $\{ \mathcal{X}(L) - \phi(a_y) : y \notin U \}$ is a cover of $\mathcal{X}(L) - U$.

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$$\mathcal{X}(L) - U \subseteq (\mathcal{X}(L) - \phi(a_{y_1})) \cup \cdots \cup (\mathcal{X}(L) - \phi(a_{y_n})).$$

Consequently $x \in \phi(a_{y_1}) \cap \cdots \cap \phi(a_{y_n}) \subseteq U$. This implies $x \in \phi(a_{y_1} \land \cdots \land a_{y_n}) \subseteq U$. 


Therefore there exists \( a = a_{y_1} \land \cdots \land a_{y_n} \) in \( L \) such that \( x \in \phi(a) \subseteq U \).
Priestley duality

Therefore there exists $a = a_{y_1} \land \cdots \land a_{y_n}$ in $L$ such that $x \in \phi(a) \subseteq U$.

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This means that $U = \bigcup\{\phi(a) : \phi(a) \subseteq U\}$. Since $U$ is closed, it is compact.
Therefore there exists $a = a_{y_1} \land \cdots \land a_{y_n}$ in $L$ such that $x \in \phi(a) \subseteq U$.

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As \( \{ \phi(a) : \phi(a) \subseteq U \} \) is an open cover of \( U \), there is a finite subcover. But a finite union of elements of the form \( \phi(a) \) is again of the same form.
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As $\{\phi(a) : \phi(a) \subseteq U\}$ is an open cover of $U$, there is a finite subcover. But a finite union of elements of the form $\phi(a)$ is again of the same form.

Therefore there is $a \in L$ such that $\phi(a) = U$ and so $\phi$ is onto.
Priestley duality

As a result, we obtain the following representation of bounded distributive lattices:
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**Priestley’s representation of bounded distributive lattices:** Each bounded distributive lattice is isomorphic to the lattice of all clopen upsets of a Priestley space.
Priestley duality

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Going the other way, we would like to show that $X$ is order-homeomorphic to $X^{**}$. 
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Going the other way, we would like to show that $X$ is order-homeomorphic to $X^{**}$. 

Define $\psi : X \rightarrow X^{**}$ by

$$\psi(x) = \{ U \in X^* : x \in U \}$$
As a result, we obtain the following representation of bounded distributive lattices:

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Going the other way, we would like to show that $X$ is order-homeomorphic to $X^{**}$.

Define $\psi : X \rightarrow X^{**}$ by

$$\psi(x) = \{ U \in X^* : x \in U \}$$

Then it is straightforward to verify that $\psi$ is well-defined.
Moreover, one can also show that $\psi$ is a continuous order-isomorphism.

This implies that each Priestley space arises up to order-homeomorphism as the Priestley space of some bounded distributive lattice. This establishes complete balance between bounded distributive lattices and Priestley spaces.

In fact it can also be extended to a complete balance between bounded lattice homomorphisms and order-preserving continuous maps. We refer to it as the Priestley duality.
Moreover, one can also show that $\psi$ is a continuous order-isomorphism. We will skip the details because this result is not absolutely necessary for our purposes.
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We only mention that since we work with compact Hausdorff spaces, it follows that $\psi : X \rightarrow X^{**}$ is an order-homeomorphism.
Moreover, one can also show that $\psi$ is a continuous order-isomorphism. We will skip the details because this result is not absolutely necessary for our purposes.

We only mention that since we work with compact Hausdorff spaces, it follows that $\psi : X \rightarrow X^{**}$ is an order-homeomorphism.

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Birkhoff’s duality as a particular case of the Priestley duality

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The Stone duality for Boolean lattices

Now suppose that $L$ is a Boolean lattice. Then it is easy to show that the poset of prime filters of $L$ is discrete.

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Conversely, we can view each Stone space $(X, \tau)$ as the Priestley space $(X, \leq, \tau)$ with the discrete $\leq$. 
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Conversely, we can view each Stone space $(X, \tau)$ as the Priestley space $(X, \leq, \tau)$ with the discrete $\leq$. Then $X^*$ becomes simply the lattice of clopen subsets of $X$, which is clearly a Boolean lattice because it is closed under set-theoretic complement.
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In particular, we obtain the following representation theorem for Boolean lattices.

**Stone’s representation of Boolean lattices:** Each Boolean lattice can be represented as the lattice of clopen subsets of a Stone space.
The Esakia duality for Heyting lattices

Let $L$ be a Heyting lattice and let $(\mathcal{X}(L), \subseteq, \tau_P)$ be the Priestley space of $L$. Then we have

$$\phi(a \rightarrow b) = \mathcal{X}(L) - \downarrow[\phi(a) - \phi(b)]$$
Let $L$ be a Heyting lattice and let $(\mathcal{H}(L), \subseteq, \tau_P)$ be the Priestley space of $L$. Then we have

$$\phi(a \to b) = \mathcal{H}(L) - \down[\phi(a) - \phi(b)]$$

Here, for any subset $S$ of a poset $P$, we denote by $\down S$ the downset of $S$:

$$\down S = \{p \in P : \exists s \in S \text{ with } p \leq s\}$$
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This allows us to give a nice characterization of dual spaces of Heyting lattices,
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Here, for any subset \( S \) of a poset \( P \), we denote by \( \downarrow S \) the **downset** of \( S \):

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This allows us to give a nice characterization of dual spaces of Heyting lattices, which was first done by Esakia in 1974.
**Theorem:** If $L$ is a Heyting lattice, then for each clopen $U$ of $L_\ast$. 

\[
\begin{tikzpicture}
    \node (U) at (0,0) {\textcolor{black}{$U$}};
    \draw [fill=black!30] (-1.5,-2) rectangle (1.5,2);
    \node (L_*) at (2,0) {\textcolor{black}{$L_\ast$}};
\end{tikzpicture}
\]
Theorem: If $L$ is a Heyting lattice, then for each clopen $U$ of $L_*$ the downset $\downarrow U$ is also clopen.
Esakia spaces

**Proof:** Let $U$ be clopen in $L_*$.
**Esakia spaces**

**Proof:** Let $U$ be clopen in $L_*$. Then there exist $a_1, \ldots, a_n, b_1, \ldots, b_n \in L$ such that

$$U = \bigcup_{i=1}^{n} (\phi(a_i) - \phi(b_i))$$
Esakia spaces

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Since $\downarrow (\phi(a_i) - \phi(b_i)) = \mathcal{X}(L) - \phi(a_i \rightarrow b_i)$,
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$$\downarrow U = \bigcup_{i=1}^{n} (\mathcal{X}(L) - \phi(a_i \rightarrow b_i))$$

Thus $\downarrow U$ is clopen.
The Esakia duality for Heyting lattices

Conversely if \((X, \leq, \tau)\) is a Priestley space in which for each clopen \(U\) we have \(\downarrow U\) is clopen,
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\[ U \rightarrow V = X - \downarrow(U - V) \]

for each \(U, V \in X^*\).
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This together with the Priestley duality establishes that there is a complete balance between Heyting lattices and those Priestley spaces in which the downset of each clopen is clopen.
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We call such spaces **Esakia spaces**.
Example

In the first lecture we had an example of a distributive lattice $L$ which is not Heyting.
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```
   1  
  /   
 a2 /     
 /   
 a1 /     
 /   
 a  
    /   
   /     
  /     
 b2     
    /     
   /     
  /     
 b1     
    /     
   /     
  /     
 0     
```

$L$
Example

In the first lecture we had an example of a distributive lattice \( L \) which is not Heyting. Thus \( L_\ast \) must fail to be an Esakia space.

\[ L = \begin{array}{c}
\text{0} \\
\text{a} \\
\text{a}_1 \\
\text{a}_2 \\
\text{1} \\
\end{array}
\]

\[ L_\ast = \begin{array}{c}
\text{\{1\}} \\
\text{\{1\}} \\
\text{\{1\}} \\
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$L$

\[
\begin{array}{c}
\downarrow a \\
0
\end{array}
\]

\[
\begin{array}{c}
\downarrow b_1 \\
\downarrow b_2 \\
\downarrow b_3 \\
\vdots
\end{array}
\]

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**Esakia’s representation of Heyting lattices:** Each Heyting lattice can be represented as the lattice of clopen upsets of an Esakia space.
Three dualities

To summarize, we have arrived at the following picture:
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Boolean lattices $\subset$ Heyting lattices $\subset$ Distributive lattices

$\leftrightarrow$

Stone spaces $\subset$ Esakia spaces $\subset$ Priestley spaces