Lattices and Topology

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Lecture 3: Topology
We have discussed the Birkhoff duality between finite distributive lattices and finite posets. Our main tools were join-prime and meet-prime elements. With each finite distributive lattice $L$ we associated its dual poset $L^*$ of join-prime elements. Conversely, with each finite poset $P$ we associated its dual lattice $P^* = \cup(P)$ of upsets. We showed that these constructions are mutually inverse in the sense that $L^{**}$ is isomorphic to $L$ and that $P^{**}$ is order-isomorphic to $P$, thus obtaining the Birkhoff duality between finite distributive lattices and finite posets.

We saw that the Birkhoff duality provides a representation of finite distributive lattices as lattices of sets with set-theoretic union and intersection.
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This forced us to generalize the concept of join-prime element to that of prime filter. Similarly, the concept of meet-prime element generalizes to that of prime ideal. We next described a representation of a (possibly infinite) distributive lattice $L$ as a sublattice of the lattice of all upset of the poset $X(L)$ of prime filters of $L$. This representation required the Stone lemma. A precise description of this sublattice can be done by topological means.
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Short outline of lecture 3

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As shown by McKinsey and Tarski in 1944, it opens the door to connect topology with modal logic.
Hausdorff’s Definition: A topological space is a pair $(X, \tau)$, where $X$ is a set and $\tau$ is a collection of subsets of $X$—called open sets—containing $\emptyset$, $X$ and closed under finite intersections and arbitrary unions.
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More precisely, $\tau$ is a topology on $X$ if

1. $\emptyset, X \in \tau$.
2. $U, V \in \tau \Rightarrow U \cap V \in \tau$.
3. $S \subseteq \tau \Rightarrow \bigcup S \in \tau$. 
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Topology

We immediately obtain:

1. $\emptyset, X \in \delta$.
2. $F, G \in \delta \implies F \cup G \in \delta$.
3. $S \subseteq \delta \implies \bigcap S \in \delta$.

Similarly to $\tau$, we have that $\delta$ is also a sublattice of $P(X)$.

Examples:

1. Let $X$ be a nonempty set. We set $\tau = P(X)$. Then $(X, \tau)$ is a topological space in which every set is open. Such spaces are called discrete spaces.
2. Now set $\tau = \{\emptyset, X\}$. Then $\tau$ is a topology on $X$, called the trivial topology.
3. Let $(P, \leq)$ be a poset. We set $\tau \leq = U(P)$. Then $(P, \tau \leq)$ is a topological space in which the intersection of any collection of opens is again open. Spaces with this property are called Alexandroff spaces.
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Interior and closure

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Let $X$ be a topological space and let $x \in X$. We call open sets containing $x$ open neighborhoods of $x$. 

Let $\text{int}(A)$ denote the interior of $A$. We also say that $x$ belongs to the closure of $A$ if each open neighborhood $U$ of $x$ has nonempty intersection with $A$. Let $\overline{A}$ denote the closure of $A$. It is easy to verify that the interior and closure satisfy the following conditions:

- $\text{int}(X) = X$
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For $A \subseteq X$, we say that $x$ belongs to the interior of $A$ if there exists an open neighborhood $U$ of $x$ contained in $A$. Let $\text{int}(A)$ denote the interior of $A$.

We also say that $x$ belongs to the closure of $A$ if each open neighborhood $U$ of $x$ has nonempty intersection with $A$. Let $\overline{A}$ denote the closure of $A$.

It is easy to verify that the interior and closure satisfy the following conditions:

- $\text{int}(X) = X$,
- $\text{int}(A) \subseteq A$,
- $\text{int}(A) \subseteq \text{int}(\text{int}(A))$,
- $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$. 
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The closure and interior operators are dual to each other:

\[ \text{int}(A) = X - \overline{X - A} \quad \text{and} \quad A = \overline{X - \text{int}(X - A)}. \]

In fact, as was shown by Kuratowski, topological spaces can be defined in terms of closure (interior) operators satisfying the above four conditions. Indeed, let:

\[ P(X) \to P(X) \]
be a function satisfying the above four conditions. Call \( A \subseteq X \) a closed subset of \( X \) if \( A = \overline{A} \).

Then \( \tau = \{ \overline{X - A} : A \text{ is closed} \} \) is a topology on \( X \), and every topology on \( X \) arises this way! The same, of course, is true if we work with \( \text{int} \): \( P(X) \to P(X) \) instead.
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Subspaces

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Continuous maps

Given two topological spaces \((X, \tau)\) and \((X', \tau')\), a map \(f : X \to X'\) is called continuous if the inverse image \(f^{-1}(U)\) of any open subset \(U\) of \(X'\) is open in \(X\).
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\[
f(t) = \frac{t}{1 - t^2}.
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Separation properties

Let $X$ be a topological space.
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either

$X$

$\cdot$

$x$

$\cdot$

$y$

$X$
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\begin{center}
\begin{tikzpicture}
  \node (X) at (0,0) {$X$};
  \node (x) at (-1,0) {$x$};
  \node (y) at (1,0) {$y$};
  \draw (x) circle (0.5cm);
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  \draw (x) -- (y);
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![Diagram](image)
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Clearly each $T_2$-space is $T_1$, and each $T_1$-space is $T_0$. 

Let $S$ be an infinite set. Then the cofinite sets of $S$ together with the empty set form a topology on $S$ we call the cofinite topology. The set $S$ with the cofinite topology is an example of a $T_1$-space which is not $T_2$.

If a set $S$ has more than one element, then the trivial topology on $S$ is not $T_0$.

All the other spaces we have considered so far are $T_0$. Any finite Hausdorff space (in fact, any finite $T_1$-space) is discrete. An example of a non-discrete Hausdorff space is the real line $\mathbb{R}$. 
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Sober spaces

Let $X$ be a topological space.
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Let $X$ be a topological space. Clearly the closures of singletons $\{x\}$ are join-prime elements of the lattice of closed subsets of $X$. On the other hand, not all $T_0$ spaces are sober. For example, the cofinite topology on an infinite set is $T_1$ but not sober. Nevertheless, one can show that each Hausdorff space is sober. An important property of sober spaces is that one can recover points of such a space from knowing only its lattice of closed sets.
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Let $X$ be a topological space. Clearly the closures of singletons $\{x\}$ are join-prime elements of the lattice of closed subsets of $X$.

We call $X$ sober if each join-prime element of the lattice of closed sets has the form $\{x\}$ for a unique point $x$. 
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Let $X$ be a $T_0$-space. We define the specialization order $\leq_{\tau}$ of $X$ by

$$x \leq_{\tau} y \iff x \in \{y\}$$

Equivalently, $x \leq_{\tau} y \iff x \in U$ implies $y \in U$ for each open set $U$.

It is easy to see that $\leq_{\tau}$ is reflexive and transitive. Moreover, $\leq_{\tau}$ is antisymmetric because $X$ is $T_0$. Note that if $X$ is $T_1$, then the specialization order is trivial.
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Equivalently $x \leq_{\tau} y$ iff $x \in U$ implies $y \in U$ for each open set $U$. Note that if $X$ is $T_1$, then the specialization order is trivial.
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Thus each $T_0$-topology $\tau$ on a set $X$ gives rise to the partial order $\leq_{\tau}$ on $X$. 

Conversely, as we already saw, each partial order $\leq$ on $X$ gives rise to the Alexandroff topology $\tau \leq$ on $X$. Therefore we obtain the following correspondences:

$$\tau \mapsto \leq \quad \tau \mapsto \leq \quad \tau \leq \tau \mapsto \leq$$

It turns out that for any partial order $\leq$ we have $\leq \tau \leq = \leq$. On the other hand $\tau \leq \tau = \tau$ iff $\tau$ is an Alexandroff topology.
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Thus each T\(_0\)-topology \(\tau\) on a set \(X\) gives rise to the partial order \(\leq_{\tau}\) on \(X\). Conversely, as we already saw, each partial order \(\leq\) on \(X\) gives rise to the Alexandroff topology \(\tau_{\leq}\) on \(X\).

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In particular, $\mathbb{R}$ and $\mathbb{Q}$ are such examples. More generally, each non-discrete (infinite) $T_1$-space is not Alexandroff.
We will need one more fundamental concept in topology, that of compactness.
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A topological space $X$ is called **compact** if for each family $\mathcal{U}$ of open subsets of $X$ with $X = \bigcup \mathcal{U}$, there exists a finite subfamily $\mathcal{U}_0 \subseteq \mathcal{U}$ such that $X = \bigcup \mathcal{U}_0$. 
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Similarly, we call a subset $C$ of $X$ compact if for each family $\mathcal{U}$ of open subsets of $X$ with $C \subseteq \bigcup \mathcal{U}$, there exists a finite subfamily $\mathcal{U}_0 \subseteq \mathcal{U}$ such that $C \subseteq \bigcup \mathcal{U}_0$. 
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(7) In fact, a subset of \(\mathbb{Q}\) is compact iff it is finite.
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**Lemma:** A closed subset of a compact space is compact.
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Proof: Let $C$ be closed and $\{U_i : i \in I\}$ be an open cover of $C$. Then $X = (X - C) \cup \bigcup_{i \in I} U_i$. Since $X$ is compact, there is a finite cover $X - C, U_1, \ldots, U_n$ of $X$. Then $U_1, \ldots, U_n$ is a finite cover of $C$. Therefore $C$ is compact.
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**Lemma:** A closed subset of a compact space is compact.

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**Lemma:** If $X$ is compact and $f : X \to Y$ is a continuous map, then $f(X)$ is compact in $Y$. 
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Compact Hausdorff spaces

An especially important subclass of the class of topological spaces is that of compact Hausdorff spaces.

- Each finite discrete space is a compact Hausdorff space.
- The interval $[0, 1]$ is a compact Hausdorff space.
- The real line $\mathbb{R}$ is Hausdorff, but it is not compact. Therefore $\mathbb{R}$ is not compact Hausdorff.

Compact Hausdorff spaces have many pleasant properties. We only mention two because they will be useful for our considerations.
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**Proof:** Let $X, Y$ be compact Hausdorff and $f : X \rightarrow Y$ be 1-1 onto continuous. The map $f$ has an inverse $g : Y \rightarrow X$; let us show that $g$ is continuous. For this it is sufficient to show that for a closed subset $F$ of $X$, $g^{-1}(F) = f(F)$ is closed.
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To single it out from the compact Hausdorff spaces, we recall that a subset of a topological space is clopen if it is both closed and open.
Stone spaces

An important subclass of the class of compact Hausdorff spaces is that of **Stone spaces** which will play a prominent role in our considerations.

To single it out from the compact Hausdorff spaces, we recall that a subset of a topological space is **clopen** if it is both **closed** and **open**.

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We call a compact Hausdorff space a Stone space if it is zero-dimensional.
Examples:
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$$X = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, 0\}$$
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\(X\) is a countably infinite Stone space.
Now we give an example of an uncountable Stone space.
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$X$ is a countably infinite Stone space. Now we give an example of an uncountable Stone space.
(3) The **Cantor set** $C$ is the closed subspace of the interval $[0, 1]$ defined as the complement of a certain union of open intervals:
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$$C = [0, 1] - \bigcup_{n=1}^{\infty} U_n,$$

where $U_1 = \left(\frac{1}{3}, \frac{2}{3}\right)$ and $U_{n+1} = \frac{1}{3}U_n \cup \left(1 - \frac{1}{3}U_n\right)$. 
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(3) The **Cantor set** $C$ is the closed subspace of the interval $[0, 1]$ defined as the complement of a certain union of open intervals:

Formally, $C$ is

$$[0, 1] - \left( \left( \frac{1}{3}, \frac{2}{3} \right) \cup \left( \frac{1}{9}, \frac{2}{9} \right) \cup \left( \frac{7}{9}, \frac{8}{9} \right) \cup \left( \frac{1}{27}, \frac{2}{27} \right) \cup \left( \frac{7}{27}, \frac{8}{27} \right) \cup \left( \frac{19}{27}, \frac{20}{27} \right) \cup \left( \frac{25}{27}, \frac{26}{27} \right) \cup \cdots \right),$$
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or, more precisely, $C = [0, 1] - \bigcup_{n=1}^{\infty} U_n,$

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(4) The interval $[0, 1]$ is a typical example of a compact Hausdorff space which is not a Stone space.