Lattices and Topology

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Lecture 2: Representation of distributive lattices

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- We have defined bounded lattices as lattices having the largest and least elements;
- We have defined complete lattices as posets whose all subsets possess glb and lub;
- We have shown that lattices can be equivalently defined as sets equipped with two binary operations $\wedge$ and $\vee$ which are idempotent, commutative, associative, and satisfy the absorption laws;

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- We have defined Boolean lattices as those distributive lattices all of whose elements have the complement;
- Finally, we have defined Heyting lattices as those distributive lattices possessing the implication for each pair of their elements.

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Posets

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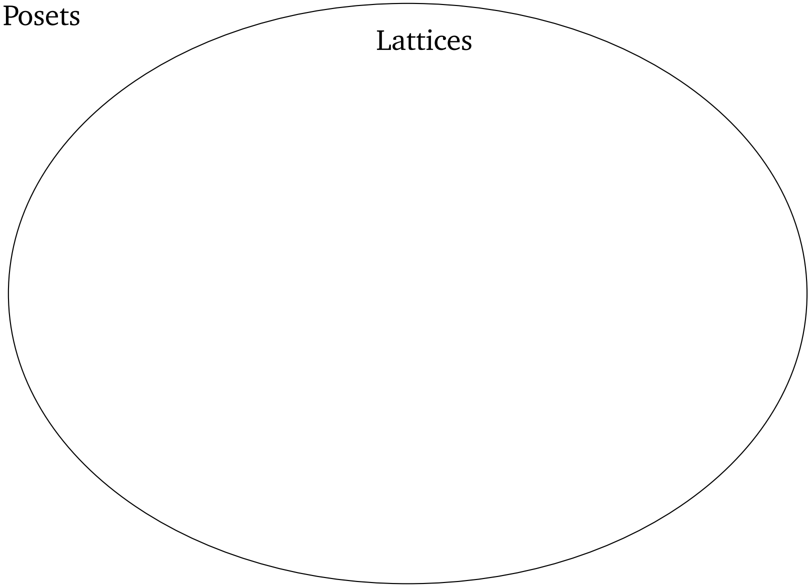

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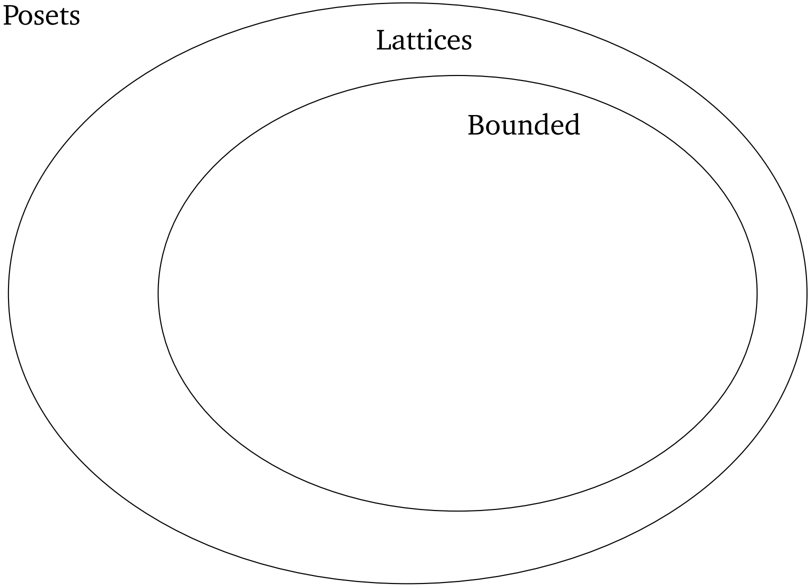

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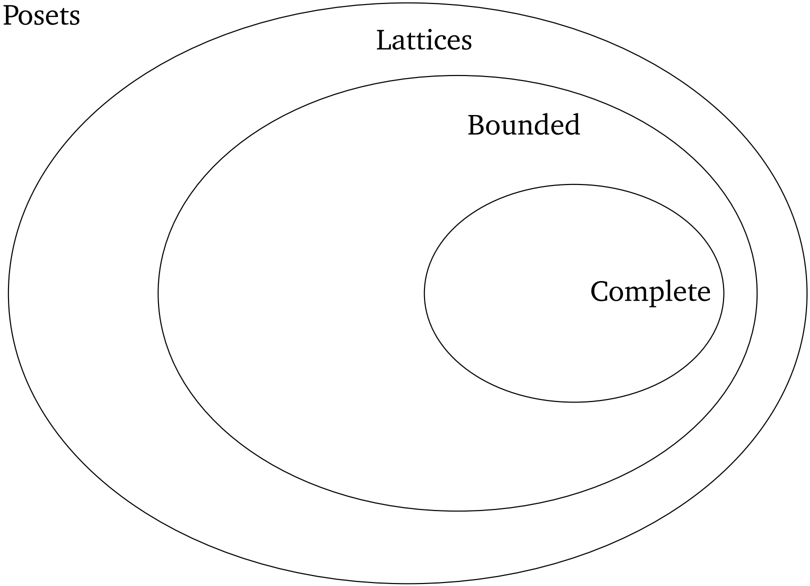

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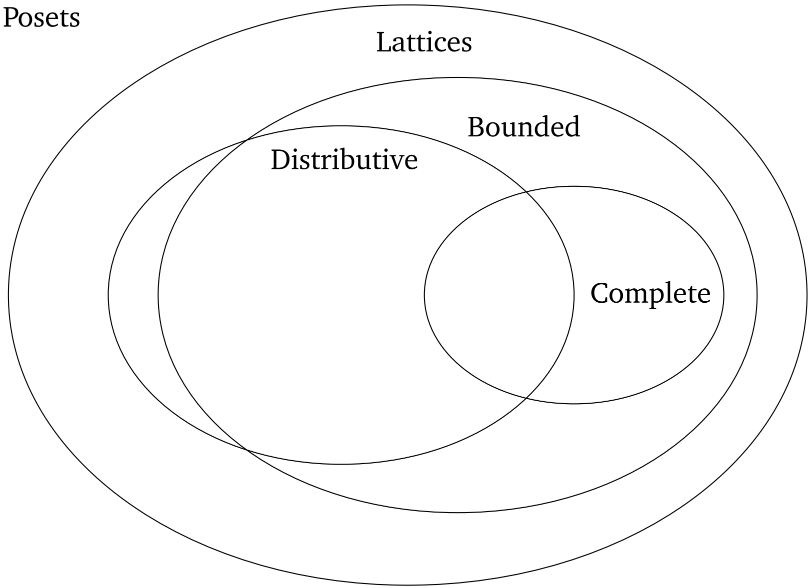

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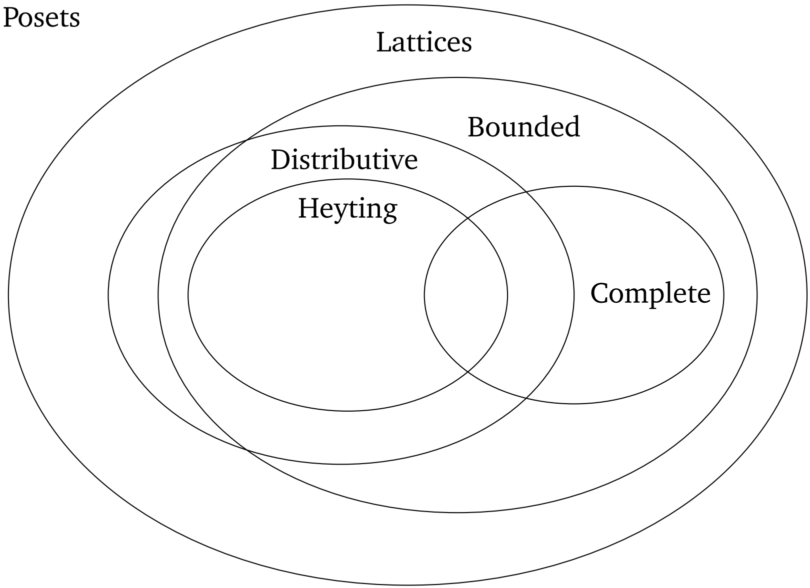

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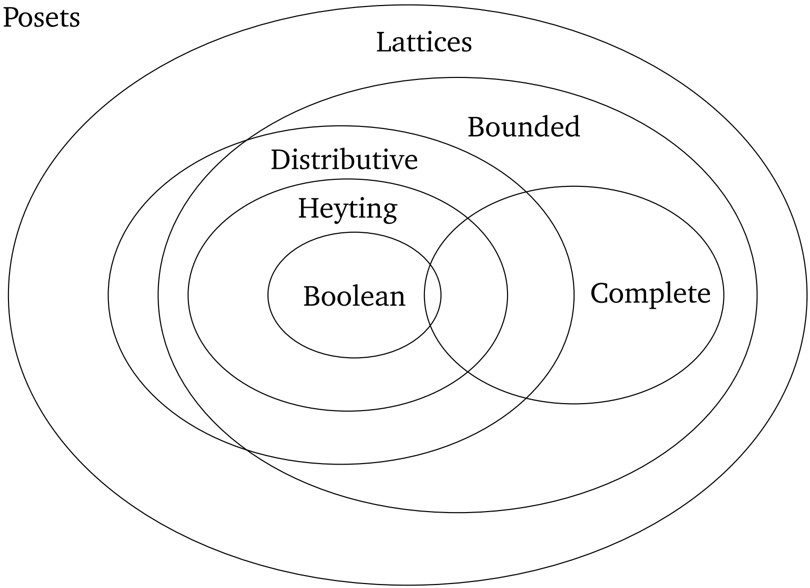

Short outline of the second lecture

Lecture 2: Representation of distributive lattices

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- Join-prime and meet-prime elements
- Birkhoff's duality between finite distributive lattices and finite posets
- Prime filters and prime ideals
- Representation of distributive lattices

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Dually, we call an element $m \neq 1$ of L meet-prime if $a \wedge b \leqslant m$ implies $a \leqslant m$ or $b \leqslant m$ for all $a, b \in L$. Let $\mathfrak{M}(L)$ denote the set of meet-prime elements of L.

Examples

In the lattice $\mathscr{U}(P)$ of upsets of a poset P, the upsets

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Similarly, in the lattice $\mathscr{D}(P)$ of downsets of P, the downsets

$$\downarrow p = \{x \in P : x \leqslant p\}$$

are join-prime for all $p \in P$.

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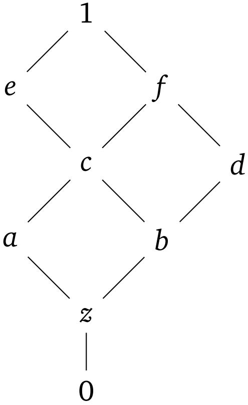

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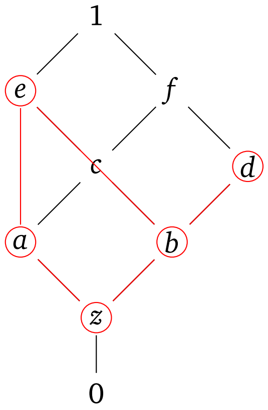

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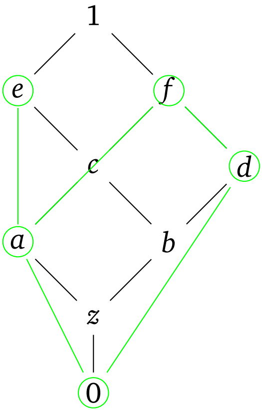

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Theorem: If L is a finite distributive lattice, then each element $a \neq 0$ of L is the join of the join-prime elements of L underneath a; that is,

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Remark: Note that all we used in the proof is that there are no infinite descending chains in L.

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Conversely, with each finite poset $(P, \leqslant)$ we associate the distributive lattice $P^* = \mathscr{U}(P)$ of upsets of P.

So we have

$$L \mapsto L_* \mapsto {L_*}^*$$

and

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That ϕ is 1-1 follows from $a = \bigvee\{j \in \mathfrak{J}(L) : j \leqslant a\}$ for each $a \in L$.

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Thus ϕ is a lattice isomorphism.

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Proof (Sketch). Define $\psi : P \to {P^*}_*$ by $\psi(p) = \uparrow p$.

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These theorems put together give us the Birkhoff duality between finite distributive lattices and finite posets.

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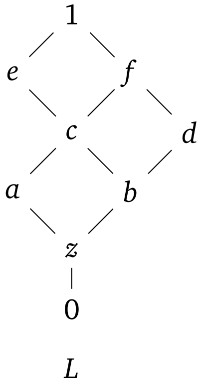

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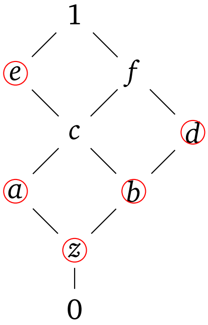

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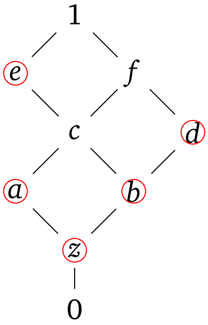

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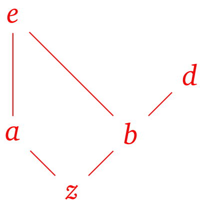

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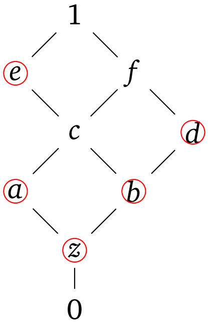

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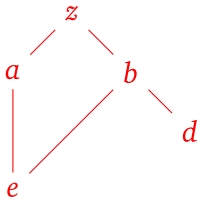

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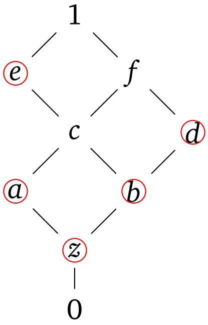

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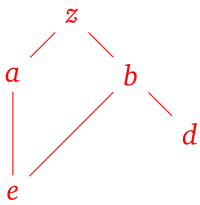

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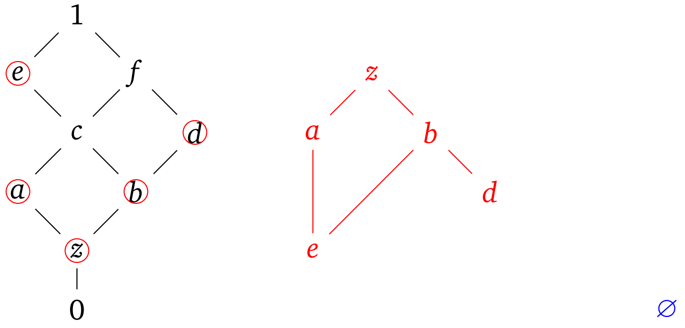

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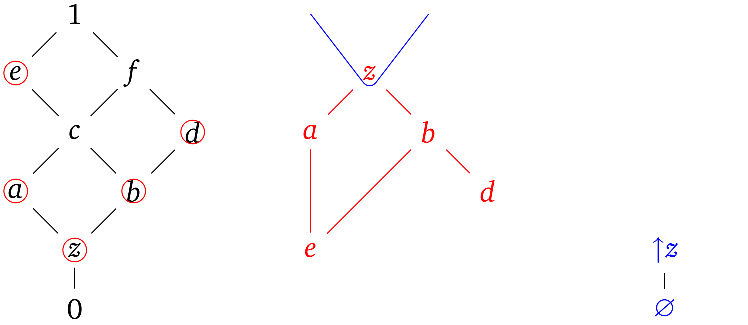

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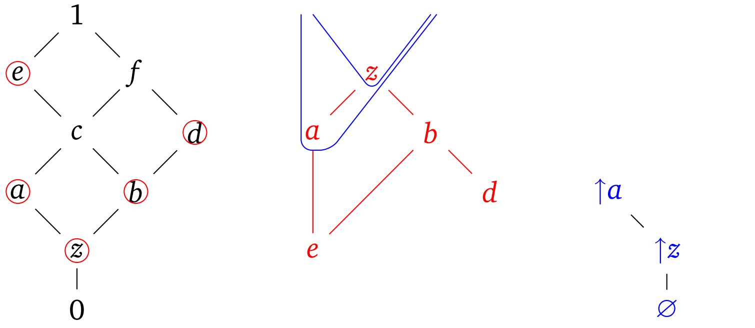

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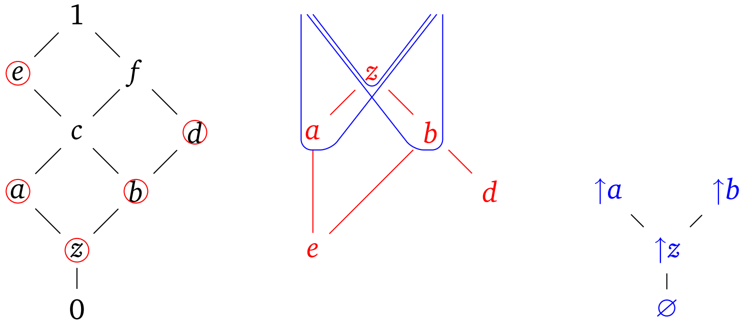

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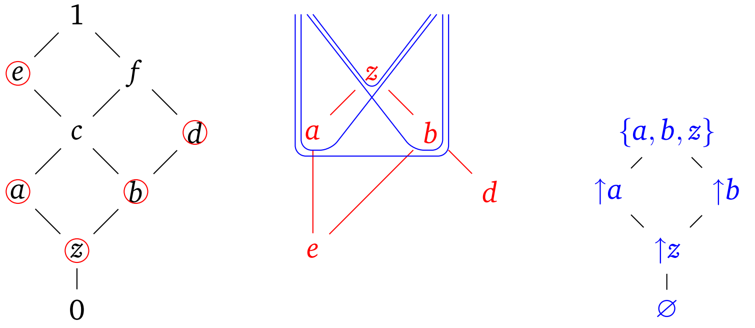

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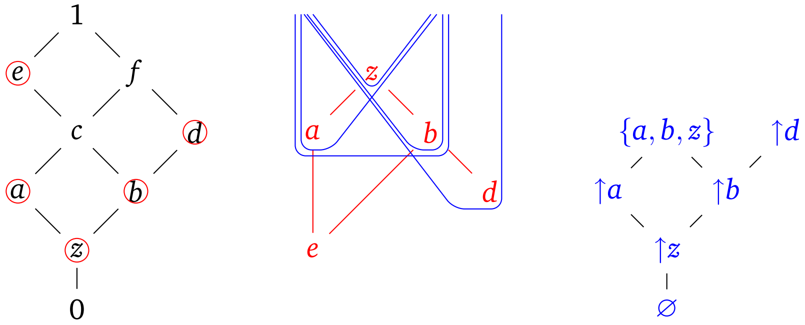

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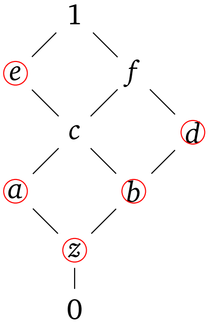

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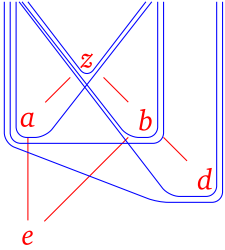

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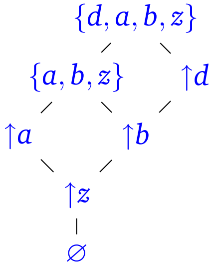

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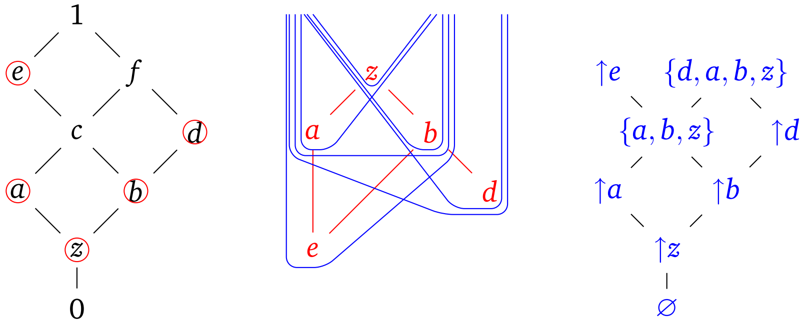

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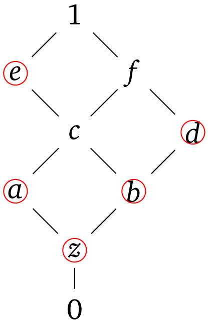

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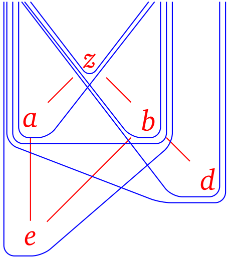

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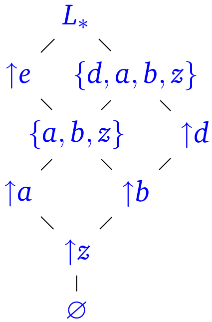

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It is our goal to extend the Birkhoff duality to all distributive lattices.

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To introduce prime filters and prime ideals, we first need to give a brief account of filters and ideals of a lattice.

The infinite case

Let L be a lattice. A nonempty subset F of L is called a filter of L if the following two conditions are satisfied:

1. From $a \in F$ and $a \leqslant b$ it follows that $b \in F$.
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A nonempty subset I of L is called an ideal of L if:

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Example: In $[0, 1]$ we have $(\frac{1}{2}, 1]$ is a non-principal filter and $[0, \frac{1}{2})$ is a non-principal ideal.

Prime filters and prime ideals

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Let L be a lattice and $F \neq L$ be a filter of L. We call F a prime filter of L if for all $a, b \in L$ we have:

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(2) Let $a \in L$. Then a is join-prime iff the principal filter $\uparrow a$ is a prime filter, and a is meet-prime iff the principal ideal $\downarrow a$ is a prime ideal.

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(1) In a linear order, every upset is a prime filter, and every downset is a prime ideal.

(2) Let $a \in L$. Then a is join-prime iff the principal filter $\uparrow a$ is a prime filter, and a is meet-prime iff the principal ideal $\downarrow a$ is a prime ideal.

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Dually, we consider the map $\mathfrak{M}(L) \to \mathscr{Y}(L)$ given by $a \mapsto \downarrow a$.

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Consequently, if L is a finite lattice, then there is an order-isomorphism between the posets $(\mathfrak{J}(L), \leqslant)$ and $(\mathfrak{M}(L), \leqslant)$.

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$$\phi(a) = \{x \in \mathscr{X}(L) \mid a \in x\}$$

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Remark: Note that $\phi(a \vee b) = \phi(a) \cup \phi(b)$ is the only place in the lemma where we require our filters to be prime!

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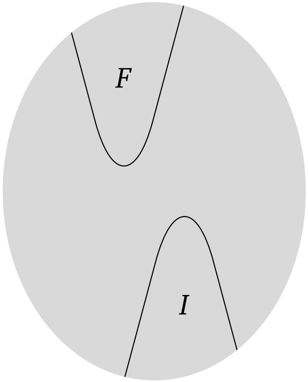
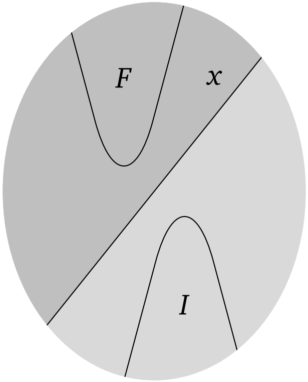


Stone's lemma

Stone's Lemma: Let L be a bounded distributive lattice, F be a filter of L and I be an ideal of L. If F and I are disjoint then there exists a prime filter x of L containing F and disjoint from I.



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As a consequence, we arrive at the following representation theorem for distributive lattices:

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Representation Theorem: Each bounded distributive lattice L is isomorphic to a sublattice of $\mathscr{U}(\mathscr{X}(L))$.

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Representation Theorem: Each bounded distributive lattice L is isomorphic to a sublattice of $\mathscr{U}(\mathscr{X}(L))$. Therefore each bounded distributive lattice can be represented as a sublattice of the lattice of upsets of some poset.

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We will outline the basic notions of topology needed for our purposes in the next lecture.