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A Brief Introduction to Topological Semantics for Modal Logic

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# Why Modal Logic

# SIMPLICITY

Simple extension of classical propositional logic by

 $\Box$  and  $\Diamond$ 

### Expressivity: many interpretations of $\Box / \Diamond$

- Necessity/Possibility
- Obligation/Permission
- 6 Kripke or relational frames
- Oescriptive frames
- Algebraic
- Interpological

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- Topological

# MODAL LOGIC LANGUAGE AND WELL FORMED FORMULAS

# Symbols

- Propositional variables/letters:  $\mathfrak{Var} = \{p_0, p_1, p_2, \dots\}$
- **2** Logical connectives:  $\top$ ,  $\perp$ ,  $\neg$ ,  $\Diamond$ ,  $\Box$ ,  $\land$ ,  $\lor$ ,  $\rightarrow$
- Ounctuation: ( and )

### WFFs

Propositional letters and ⊤ and ⊥
 If φ, ψ are WFF then so are
 (¬φ), (◊φ), (□φ)
 (φ ∧ ψ), (φ ∨ ψ), (φ → ψ)
 NOTE: Drop parenthesis–Unary connectives bind closer than binary;
 e.g. write □p → p for ((□p) → p)

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- $\textcircled{O} \text{ Logical connectives: } \top, \ \bot, \ \neg, \ \Diamond, \ \Box, \ \wedge, \ \lor, \ \rightarrow$
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## WFFs

 ${\small {\small \bigcirc }} {\small {\rm Propositional \ letters \ and \ } \top \ and \ \bot }$ 

**2** If  $\varphi, \psi$  are WFF then so are  $(\neg \varphi), (\Diamond \varphi), (\Box \varphi)$ 

$$(\varphi \wedge \psi), \ (\varphi \lor \psi), \ (\varphi \to \psi)$$

NOTE: Drop parenthesis–Unary connectives bind closer than binary; e.g. write  $\Box p \rightarrow p$  for  $((\Box p) \rightarrow p)$ 

# MODAL LOGIC BASIC AXIOMS AND INFERENCE RULES

## DEFINITION

- L is a (normal) modal logic if L contains:
  - $\textcircled{ O Classical tautologies: e.g. } p \lor \neg p \text{ and } p \to (q \to p)$

$$\textcircled{0} \hspace{0.1in} \mathsf{K} = \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$$

and L is closed under the inference rules:

- Modus Ponens (MP)
- Substitution (Sub)
- Inecessitation (N)

$$(\mathsf{MP}) \quad \frac{\varphi, \varphi \to \psi}{\psi} \qquad (\mathsf{SUB}) \quad \frac{\varphi(p)}{\varphi(\psi)} \qquad (\mathsf{N}) \quad \frac{\varphi}{\Box \varphi}$$

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- ② K4 = K +  $\Diamond \Diamond p \rightarrow \Diamond p$ logic of transitive ( $\forall w \forall v \forall u \ w Rv \& vRu \Rightarrow wRu$ ) frames
- S4 = K4 + p → ◊p logic of reflexive (∀w wRw) and transitive frames (a.k.a. quasi-order or preorder)
- **K4D** = **K4** +  $\Diamond$ T logic of serial ( $\forall w \exists v \ w Rv$ ) transitive fra

### Equivalent Formulas

• 
$$\Box(p 
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## Equivalent Formulas

$$\begin{array}{c} \square p \leftrightarrow \neg \Diamond \neg p \\ \Diamond p \leftrightarrow \neg \square \neg p \end{array}$$

# TOPOLOGICAL SPACES

# DEFINITION

Call  $(X, \tau)$  a topological space if  $\tau \subseteq \mathcal{P}(X)$  $X, \varnothing \in \tau$ 

$$U, V \in \tau \Rightarrow U \cap V \in \tau$$

$$U_i \in \tau \Rightarrow \bigcup_{i \in I} U_i \in \tau \text{ for any indexing set } I$$

U open:  $U \in \tau$ C closed:  $X - C \in \tau$ 

#### Recall

For  $A \subseteq X$  there are

INTERIOR The greatest open set contained in A, int(A)CLOSURE The least closed set containing A,  $\overline{A}$ 

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# PROPERTIES OF INTERIOR AND CLOSURE

## SET OPERATIONS

$$nt(A) = \bigcup \{ U \in \tau : U \subseteq A \}$$
  
$$\overline{A} = \bigcap \{ C : A \subseteq C \text{ and } X - C \in \tau \}$$

#### CHARACTERIZATION

L

 $\begin{array}{ll} x \in int(A) & \text{iff} \quad \exists U \in \tau, \; x \in U \; \text{and} \; \forall y \in U, \; y \in A \\ x \in \overline{A} & \text{iff} \quad \forall U \in \tau, \; x \in U \Rightarrow \exists y \in U, \; y \in A \end{array}$ 

$$int(A) = X - \overline{X - A}$$
  
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# More Properties of Interior and Closure

Set Inclusions					
int(A)	$\subseteq$	A	A	$\subseteq$	Ā
X	$\subseteq$	int(X)	$\overline{\varnothing}$	$\subseteq$	Ø
int(A)	$\subseteq$	int(int(A))	$\overline{\overline{A}}$	$\subseteq$	$\overline{A}$
$int(A \cap B)$	=	$int(A) \cap int(B)$	$\overline{A \cup B}$	=	$\overline{A} \cup \overline{B}$

### VALUATIONS

# A function $\nu : \mathfrak{Var} \to \mathcal{P}(X)$ is a valuation

Intuitively,  $\nu$  indicates where each WFF,  $\varphi$  is true Formally, for  $x \in X$ 



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$$\begin{array}{lll} x \models p & \text{iff} & x \in \nu(p) \\ x \models \neg \varphi & \text{iff} & x \not\models \varphi \\ x \models \varphi \land \psi & \text{iff} & x \models \varphi \text{ and } x \models \psi \\ x \models \Box \varphi & \text{iff} & \exists U \in \tau, \ x \in U \text{ and } \forall y \in U, \ y \models \varphi \end{array}$$
  
Hence,  $x \models \Diamond \varphi & \text{iff} & \forall U \in \tau, \ x \in U \Rightarrow \exists y \in U, \ y \models \varphi \end{array}$ 

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### Observation: $\varphi$ defines a subset of X

Given  $\nu$ , Let  $||\varphi|| = \{x \in X : x \models \varphi\}$ . Then

$  \Diamond \varphi  $	$  \varphi  $
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#### VALID FORMULAS

- $\varphi$  is valid in X provided  $\forall \nu, \forall x \in X, x \models \varphi$ ; write  $X \models \varphi$ Equivalently  $||\varphi|| = X$  for each  $\nu$
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- ② For a class of spaces C, L<sub>c</sub>(C) = {φ : ∀X ∈ C, X ⊨ φ} is a modal logic (Exercise)
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### Observation: $\varphi$ defines a subset of X

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- ② For a class of spaces C, L<sub>c</sub>(C) = {φ : ∀X ∈ C, X ⊨ φ} is a modal logic (Exercise)
- $L_c(X) = \{ \varphi : X \models \varphi \}$  in case C is only one space.

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# EXPRESSIVITY AND A BASIC RESULT

## Formulas and Properties

$$p \to \Diamond p \qquad A \subseteq \overline{A}$$
  

$$\Diamond \bot \to \bot \qquad \overline{\varnothing} \subseteq \varnothing$$
  

$$\Diamond \Diamond p \to \Diamond p \qquad \overline{\overline{A}} \subseteq \overline{A}$$
  

$$\Diamond (p \lor q) \leftrightarrow (\Diamond p \lor \Diamond q) \qquad \overline{A \cup B} = \overline{A} \cup \overline{B}$$

#### $\Gamma$ heorem

Let **Top** be class of all topological spaces,  $L_c(Top) = S4$ .

 $L_c(Top) ⊇ S4 (Sound)$  $L_c(Top) ⊆ S4 (Complete)$ 

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### Theorem

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 $L_c(Top) \supseteq S4$  $L_c(Top) \subseteq S4$ 

(Complete)

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# EXPRESSIVITY AND A BASIC RESULT

## Formulas and Properties

$$p \to \Diamond p \qquad A \subseteq \overline{A}$$
$$\Diamond \bot \to \bot \qquad \overline{\varnothing} \subseteq \varnothing$$
$$\Diamond \Diamond p \to \Diamond p \qquad \overline{\overline{A}} \subseteq \overline{A}$$
$$\Diamond (p \lor q) \leftrightarrow (\Diamond p \lor \Diamond q) \qquad \overline{A \cup B} = \overline{A} \cup \overline{B}$$

### Theorem

Let **Top** be class of all topological spaces,  $L_c(Top) = S4$ .

$$\begin{array}{ll} \mathsf{L}_c(\mathsf{Top}) \supseteq \mathsf{S4} & (\mathsf{Sound}) \\ \mathsf{L}_c(\mathsf{Top}) \subseteq \mathsf{S4} & (\mathsf{Complete}) \end{array}$$

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# Specialization Order and S4-frames

### DEFINITION

Let  $(X, \tau) \in \mathbf{Top}$ Put  $xR_{\tau}y$  iff  $x \in \overline{\{y\}}$ Call  $R_{\tau}$  the specialization order on X (generated by  $\tau$ )

#### BASIC RESULTS (EXERCISES)

```
• R_{	au} is a quasi-order
```

② If au is  $\mathsf{T}_1$  (points are closed) then  $R_ au = \{(x,x): x \in X\}$ 

#### Examples: au to $R_{ au}$

- Two point spaces: trivial, Sierpinski, discrete ((R<sub>τ</sub>)<sup>-1</sup> is closure)
- 2 Real line  $\mathbb{R}$   $((R_{\tau})^{-1}$  is not closure)

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- $R_{\tau}$  is a quasi-order
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# Specialization Order and $\textbf{S4}\text{-}\textsc{frames}_{\textsc{ii}}$

## DEFINITION

Let (W, R) be a quasi-order (reflexive and transitive) Call  $U \subseteq W$  an R-upset if  $w \in U$  &  $wRv \Rightarrow v \in U$ 

Call  $\tau_R$  the Alexandrov topology on W (generated by R)

#### Examples: R to $au_R$

**()** Two point frames: cluster, chain, anti-chain (closure is  $R^{-1}$ )

- 2 Two Fork (closure is  $R^{-1}$ )
- ${old 0}~({\mathbb R},\leq)$  (closure is  $\leq^{-1})$

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#### EXAMPLES: R to $\tau_R$

• Two point frames: cluster, chain, anti-chain (closure is  $R^{-1}$ )

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- $(\mathbb{R}, \leq)$  (closure is  $\leq^{-1}$ )

# Specialization Order and $\textbf{S4}\text{-}\textsc{frames}_\textsc{iii}$

## BASIC RESULTS (EXERCISES)

- $\tau_R$  is a topology satisfying  $U_i \in \tau_R \Rightarrow \bigcap_{i \in I} U_i \in \tau_R$
- $In (W, \tau_R), \overline{A} = R^{-1}(A) = \{ w \in W : \exists v \in A \ wRv \}$
- If  $R = \{(w, w) : w \in W\}$  then  $\tau_R = \mathcal{P}(W)$
- *R* is partial order iff  $\tau_R$  is T<sub>0</sub>
  - Partial order is a quasi-order that is antisymmetric (∀w∀v wRv & vRw ⇒ w = v)
  - In a T<sub>0</sub> space for each pair of distinct points there is an open set that contains exactly one of the pair (∀x∀y, ∃U ∈ τ, x ∈ U & y ∉ U or x ∉ U & y ∈ U)

# Specialization Order and ${\bf S4}\xspace$ -frames $\xspace{\rm III}$

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## DEFINITION

Call  $(X, \tau) \in$  **Top** an Alexandrov space provided

$$U_i \in \tau \Rightarrow \bigcap_{i \in I} U_i \in \tau$$
 for any indexing set  $I$ 

E.g. For (W, R) a quasi-order,  $(W, \tau_R)$  is an Alexandrov space Let **Alex** be the class of all Alexandrov spaces

#### Theorems (exercise)

$$\textbf{0} \ (X,\tau) \in \textbf{Alex} \text{ iff } \forall x \in X \text{ there is least } U \in \tau \text{ with } x \in U$$

② 
$$R=R_{ au_R}$$
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# C-SEMANTICS REALIZING KRIPKE SEMANTICS

#### Theorem

Let (W, R) be a quasi-order

 $(W, R) \models \varphi \text{ iff } (W, \tau_R) \models \varphi$ 

• Frame semantics for quasi-orders is special case of c-semantics So frame completeness moves to topological completeness

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$$L_c(Alex) = S4$$

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### Theorem

• 
$$L_c(Top) = S4$$

2 Let X be (separable) metrizable dense-in-itself space,
 L<sub>c</sub>(X) = S4
 dense-in-itself: X has no isolated points, that is {x} ∉ c

# **3** $\mathsf{L}_c(\mathbb{R}^2) = \mathsf{L}_c(\mathbb{R}) = \mathsf{L}_c(\mathbb{Q}) = \mathsf{L}_c(\mathsf{C}) = \mathsf{S}^4$

#### Remark

Idea is to move frame completeness to topological completeness via functions that make  $R^{-1}$  coincide with closure; i.e.

$$f^{-1}(R^{-1}(A)) = \overline{f^{-1}(A)}$$

Such functions are called interior functions; some examples for  $\mathbb R$ 

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# TOPOLOGICAL DERIVATIVE LIMIT POINT OPERATOR

# Recall

Definition: For  $A \subseteq X$ ,

 $x \in d(A)$  iff  $\forall U \in \tau, x \in U \Rightarrow \exists y \in U - \{x\}, y \in A$ 

Properties:

 $\overline{A} = A \cup d(A)$  $d(\emptyset) \subseteq \emptyset$  $d(A \cup B) = d(A) \cup d(B)$  $d(d(A)) \subseteq A \cup d(A)$ 

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# TOPOLOGICAL DERIVATIVE DUAL OPERATOR

## DEFINITION

Coderivative t is dual to derivative; so ...

t(A) = X - d(X - A) $x \in t(A)$  iff  $\exists U \in \tau, \ x \in U \& \ \forall y \in U - \{x\}, \ y \in A$ 

Also

$$d(A) = X - t(X - A)$$
  

$$int(A) = A \cap t(A)$$
  

$$t(X) \supseteq X$$
  

$$t(A \cap B) = t(A) \cap t(B)$$
  

$$A \cap t(A) \subseteq t(t(A))$$

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$$t(A) = X - d(X - A)$$
  
  $x \in t(A)$  iff  $\exists U \in \tau, x \in U \& \forall y \in U - \{x\}, y \in A$ 

Also

$$d(A) = X - t(X - A)$$
  

$$int(A) = A \cap t(A)$$
  

$$t(X) \supseteq X$$
  

$$t(A \cap B) = t(A) \cap t(B)$$
  

$$A \cap t(A) \subseteq t(t(A))$$

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# TOPOLOGICAL DERIVATIVE DUAL OPERATOR

## DEFINITION

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# DIAMOND AS DERIVATIVE

# VALUATIONS

A valuation is  $\nu : \mathfrak{Var} \to \mathcal{P}(X)$ 

$$\begin{array}{lll} x \models p & \text{iff} & x \in \nu(p) \\ x \models \neg \varphi & \text{iff} & x \not\models \varphi \\ x \models \varphi \land \psi & \text{iff} & x \models \varphi \text{ and } x \models \psi \\ x \models \Box \varphi & \text{iff} & \exists U \in \tau, \ x \in U \text{ and } \forall y \in U - \{x\}, \ y \models \varphi \\ \text{Hence,} \\ x \models \Diamond \varphi & \text{iff} & \forall U \in \tau, \ x \in U \Rightarrow \exists y \in U - \{x\}, \ y \models \varphi \end{array}$$

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# DIAMOND AS DERIVATIVE

### As Before: $\varphi$ defines a subset of X

Given  $\nu$ , Put  $||\varphi|| = \{x \in X : x \models \varphi\}$ . Then  $||\Diamond \varphi|| = d(||\varphi||)$ 

$$||\Box\varphi|| = t(||\varphi||)$$

#### VALIDITY

- $\varphi$  is valid in X provided  $\forall \nu, ||\varphi|| = X$
- L<sub>d</sub>(C) = {φ : ∀X ∈ C, X ⊨ φ} is a modal logic for any class of spaces C (Exercise)
- If  $L = L_d(C)$  for some class C of spaces, call L a d-logic
- If  $L = L_c(C)$  for some class C of spaces, call L a c-logic
# DIAMOND AS DERIVATIVE

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#### VALIDITY

- $\varphi$  is valid in X provided  $\forall \nu$ ,  $||\varphi|| = X$
- Solution 2  $\{\varphi : \forall X \in \mathcal{C}, X \models \varphi\}$  is a modal logic for any class of spaces  $\mathcal{C}$  (Exercise)

 $||\Box \varphi|| = t(||\varphi||)$ 

- ③ If  $\mathsf{L} = \mathsf{L}_d(\mathcal{C})$  for some class  $\mathcal{C}$  of spaces, call  $\mathsf{L}$  a d-logic
- ① If  $\mathsf{L} = \mathsf{L}_c(\mathcal{C})$  for some class  $\mathcal{C}$  of spaces, call  $\mathsf{L}$  a c-logic

# DIAMOND AS DERIVATIVE

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### Some Topological Properties

#### RECALL (EXERCISES)

### (X, τ) is dense-in-itself (dii) if X has no isolated points (∀x ∈ X, {x} ∉ τ) Equivalently...

$$d(X) = X$$

② (X, τ) is T<sub>d</sub> provided points are locally closed (∀x ∈ X ∃U ∈ τ, {x} = U ∩ {x}) Equivalently... ∀A ⊆ X,

$$d(d(A)) \subseteq d(A)$$

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### EXPRESSIVITY IN D-SEMANTICS

#### FORMULAS AND PROPERTIES

Always Valid:		
$\Diamond\bot\to\bot$	$d(arnothing)\subseteq arnothing$	
$\Diamond\Diamond p  o p \lor \Diamond p$	$d(d(A))\subseteq A\cup d(A)$	
$\Diamond(p\lor q)\leftrightarrow(\Diamond p\lor\Diamond q)$	$d(A\cup B)=d(A)\cup d(B)$	
Sometimes Valid:		
$\Diamond\Diamond ho ho ightarrow \Diamond ho$	$d(d(A)) \subseteq d(A)$	$(T_d)$
$\Diamond\top$	d(X) = X	(dii)
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So d-semantics is strictly more expressive than c-semantics!

#### Theorem

**2** 
$$L_d({X \in \text{Top} : X \text{ is } T_d}) = K4$$

•  $L_d({X \in \text{Top} : X \text{ is dii and } T_d}) = K4D$ 

#### As Before:

Utilize results in frame semantics But the new situation is more delicate Recall closure 'was'  $R^{-1}$ ... Want similar for d

#### EXAMPLES:

- 2 point spaces: trivial and Sierpinski
- ② Distinguish between line and plane

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#### DEFINITION

Call (W, R) weakly transitive if  $\forall w \forall v \forall u \ w Rv \ \& \ v Ru \ \& \ w \neq u \Rightarrow w Ru$ 

#### LEMMA (Exercise)

(W,R) is weakly transitive iff  $(W,R) \models \Diamond \Diamond p \rightarrow (p \lor \Diamond p)$ 

#### DEFINITION

For  $(X, \tau) \in$  **Top**, put  $xS_{\tau}y$  iff  $x \in d(\{y\})$ 

- (X, S<sub>τ</sub>) is weakly transitive and irreflexive (no point is related to itself)
- $\ \, {\it O} \ \, S_\tau = R_\tau \{(x,x): x\in X\} \ \, ({\it recall} \ \, R_\tau \ \, {\it is specialization order})$

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#### BASIC RESULTS (EXERCISES)

(X, S<sub>τ</sub>) is weakly transitive and irreflexive (no point is related to itself)

②  $S_ au = R_ au - \{(x,x): x \in X\}$  (recall  $R_ au$  is specialization order)

#### DEFINITION

Call (W, R) weakly transitive if  $\forall w \forall v \forall u \ w Rv \ \& \ v Ru \ \& \ w \neq u \Rightarrow w Ru$ 

#### LEMMA (EXERCISE)

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For 
$$(X, \tau) \in$$
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- (X, S<sub>τ</sub>) is weakly transitive and irreflexive (no point is related to itself)
- **2**  $S_{ au} = R_{ au} \{(x,x) : x \in X\}$  (recall  $R_{ au}$  is specialization order)

# Analogue to Specialization Order

# MORE BASIC RESULTS (EXERCISES) $S_{\tau_{R}} \subseteq R$ • $\tau \subseteq \tau_{S_{-}}$ • $\tau_{S_{\pi}}$ is Alexandrov topology • $S_{\tau_B} = R$ • $d(A) = R^{-1}(A)$ in $(W, \tau_R)$

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# Analogue to Specialization Order $_{\rm II}$

#### More Basic Results (Exercises)

- $I S_{\tau_R} \subseteq R$
- $\tau_{S_{\tau}} = \tau_{R_{\tau}}$ Hence...
  - $\tau \subseteq \tau_{S_{\tau}}$
  - $\tau_{\mathcal{S}_{\tau}}$  is Alexandrov topology

If R is irreflexive and weakly transitive then

• 
$$S_{\tau_R} = R$$
  
•  $d(\Lambda) = R^{-1}(\Lambda)$  in (M

• 
$$d(A) = R^{-1}(A)$$
 in  $(W, \tau_R)$ 

• If X is Alexandrov then  $au = au_{S_{ au}}$ 

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#### Theorem

Let (W, R) be an irreflexive weakly transitive frame In d-semantics

 $(W, R) \models \varphi \text{ iff } (W, \tau_R) \models \varphi$ 

- Frame semantics for irreflexive weakly transitive frames is special case of d-semantics
- L<sub>d</sub>(Alex) = wK4 (Note: wK4 is logic of irreflexive weakly transitive frames)
- $L_d(Alex_{fin}) = wK4$

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#### Theorem

For a separable metrizable dense-in-itself 0-dimensional space
 X, L<sub>d</sub>(X) = K4D

0-dimensional: clopens form basis for au

• 
$$L_d(\mathbb{R}^2) = K4D + G_1$$
 where

$$\mathsf{G}_1 = (\Diamond p \land \Diamond \neg p) \to \Diamond ((p \lor \Diamond p) \land (\neg p \lor \Diamond \neg p))$$

$$\bullet \ \mathsf{L}_d(\mathbb{R}) = \mathsf{K4D} + \mathsf{G}_2$$

#### Remark

As before, move frame completeness to d-semantics via functions

$$f^{-1}(R^{-1}(A)) = d(f^{-1}(A))$$

# Completeness in d-Semantics

#### THEOREM

For a separable metrizable dense-in-itself 0-dimensional space X, L<sub>d</sub>(X) = K4D
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